Fourier Analysis

Workshop 9: More on Partial Differential Equations

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1. Find solutions $u(x,y)$ by separation of variables to
(a)

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$$

(b)

$$\frac{\partial u}{\partial x} - xy\frac{\partial u}{\partial y} = 0$$

2. Consider a particle of mass m which is confined within a square well $0 < x < \pi$, $0 < y < \pi$. The steady-state 2D Schrödinger equation inside the well (where the potential is zero) is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi.$$

The walls have infinite potential, so $\psi = 0$ on the boundaries.
(a) Find separable solutions $\psi(x,y) = X(x)Y(y)$ and show that they are

$$\psi(x,y) = A\sin(rx)\sin(ny)$$

for integers r, n.

(b) The wavefunction is normalised so that $\int |\psi(x,y)|^2\,dx\,dy = 1$. For given r, n, find A.

(c) Show that the energy levels corresponding to the quantum numbers m, n are

$$E = (r^2 + n^2)\frac{\hbar^2}{2m}.$$

3. Show by direct substitution into the equation that

$$u(x,t) = f(x - ct) + g(x + ct)$$

where f and g are arbitrary functions, is a solution of the 1D wave equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where the sound speed c is a constant. You may recall that the partial derivative of $f(y)$ with respect to x (where y may be a function of several variables $y(x,t,\ldots)$) is

$$\frac{\partial f}{\partial x} = \frac{\partial y}{\partial x}\frac{df}{dy}.$$

4. (a) Consider separable solutions for the temperature $u(x,t) = X(x)T(t)$ of the 1D heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

and find the differential equations which X and T must satisfy, giving your reasoning.

(b) Solving for T, show that the separable solutions which are finite as $t \to \infty$ are of the form

$$[A\cos(kx) + B\sin(kx)]\exp(-k^2 t).$$

where $k^2 > 0$.

(c) There is one more (rather simple) permitted solution. What is it?

(d) Following on from the last question, find all solutions for which $u(0,t) = u(\pi,t) = 0$ at all times. Hint: the answer is not a single term, but rather a sum.

(e) If the initial temperature (at $t = 0$) is $u(x,0) = \sin x \cos x$, what is the full solution $u(x,t)$?

5. This is a question which looks hard, because it uses polar coordinates, but you can solve it in exactly the same way as the Cartesian equations.

Laplace's equation in polar coordinates r, θ is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

(a) Show that for solutions which are separable, $u(r,\theta) = R(r)\Theta(\theta)$,

$$\Theta''(\theta) = -k^2\Theta; \qquad r^2 R''(r) + rR'(r) - k^2 R(r) = 0$$

for some constant k^2.

(b) Argue that the solution must be periodic in θ, and say what the period must be.

(c) As a consequence, what values of k are permitted?

(d) By trying power-law solutions $R(r) \propto r^\alpha$, find the general solution which is finite at the origin.

(e) Find the solution for a situation where u is fixed on a circular ring at $r = 1$ to be

$$u(r = 1, \theta) = \sin^2\theta + 2\sin\theta\cos\theta.$$