Fourier Analysis

Workshop 8: Fourier solutions of Partial Differential Equations

Professor John A. Peacock
School of Physics and Astronomy
jap@roe.ac.uk
Session: 2014/15
11th & 14th November 2014

1. If we have a function $u(x,t)$, we may do a partial Fourier Transform, changing x to k but leaving t in the equations. Ths approach is useful because of the following result:

$$FT\left[\frac{\partial u(x,t)}{\partial t}\right] = \frac{\partial \tilde{u}(k,t)}{\partial t}.$$

Show this (you can probably fit it on one line).

2. Consider the 1D wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

with boundary conditions at $t=0$ that $u(x,t) = e^{-a|x|}$ for some $a>0$, and $\partial u(x,t)/\partial t = 0$.
(a) By applying a Fourier Transform with respect to x, show that the FT of the general solution is of the form

$$\tilde{u}(k,t) = A(k)e^{-ikct} + B(k)e^{ikct}.$$

(b) Show that at $t=0$,

$$\tilde{u}(k,0) = \frac{2a}{a^2+k^2}.$$

(c) Hence, applying the boundary conditions, show that

$$\tilde{u}(k,t) = \frac{a}{a^2+k^2}\left(e^{-ickt} + e^{ickt}\right).$$

Note that you will need to argue that the boundary condition on $\partial u/\partial t$ also applies to each Fourier component individually.

(d) Finally deduce that

$$u(x,t) = \frac{1}{2}\left(e^{-a|x-ct|} + e^{-a|x+ct|}\right).$$

3. Consider the 1D diffusion equation for the temperature $u(x,t)$,

$$\frac{\partial^2 u}{\partial x^2} = \kappa \frac{\partial u}{\partial t},$$

where the initial condition is that $u(x, t=0) = \phi(x)$.

(a) Take the Fourier Transform with respect to x, i.e.

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx.$$

Note that the transform is still a function of t. Show that it obeys

$$\frac{\partial \tilde{u}(k,t)}{\partial t} = -\frac{k^2}{\kappa} \tilde{u}(k,t).$$

(b) Treating k as a constant so far as the time coordinate is concerned, use an integrating factor to find

$$\tilde{u}(k,t) = \tilde{f}(k) e^{-k^2 t/\kappa},$$

for some (arbitrary) function $\tilde{f}(k)$.

(c) From the initial condition $u(x,0) = \phi(x)$ show that

$$\tilde{f}(k) = \tilde{\phi}(k) \quad \Rightarrow \quad \tilde{u}(k,t) = \tilde{\phi}(k) e^{-k^2 t/\kappa}.$$

(d) Using the result that the FT of $e^{-\kappa x^2/(4t)}$ is $\sqrt{4\pi t/\kappa}\, e^{-k^2 t/\kappa}$, show using the convolution theorem that the general solution for $u(x,t)$ in terms of $\phi(x)$ is

$$u(x,t) = \frac{\sqrt{\kappa}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\kappa (x-x')^2/(4t)} \phi(x')\, dx'.$$

(e) If $\phi(x) = \delta(x-1)$, what is $u(x,t)$?

4. The charge density ρ and the electrostatic potential Φ are related by Poisson's equation

$$\nabla^2 \Phi(x) = -\frac{\rho(x)}{\epsilon_0},$$

where we assume that there is no time dependence. Treating this as a one-dimension problem (so $\nabla^2 \to d^2/dx^2$), show using a Fourier Transform that a Gaussian potential

$$\Phi(x) = e^{-x^2/(2\sigma^2)}$$

is sourced by a charge density field

$$\rho(x) = \frac{\epsilon_0}{\sigma^2}\, e^{-x^2/(2\sigma^2)} \left(1 - \frac{x^2}{\sigma^2}\right).$$

In doing this, you will demonstrate that

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 e^{-k^2\sigma^2/2} e^{ikx} = \frac{1}{\sqrt{2\pi}\sigma^3}\, e^{-x^2/(2\sigma^2)} \left(1 - \frac{x^2}{\sigma^2}\right).$$

Verify the solution by direct differentiation of $\Phi(x)$.

You may assume that the Fourier Transform of $e^{-x^2/(2\sigma^2)}$ is $\sqrt{2\pi}\sigma\, e^{-k^2\sigma^2/2}$, and that $\int_{-\infty}^{\infty} e^{-u^2/2} du = \sqrt{2\pi}$. (This method is more often used in reverse, where ρ is known and you want Φ).

5. [Hint: do all of this in Cartesian coordinates – do not be tempted to use spherical polars, despite the symmetry of the problem].

The charge density ρ and the electrostatic potential Φ are related by Poisson's equation

$$\nabla^2\Phi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon_0},$$

where we assume that there is no time dependence. Treating this now as a 3D problem (so $\nabla^2 \to \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$), show using a Fourier Transform that a Gaussian potential

$$\Phi(\boldsymbol{r}) = e^{-r^2/(2\sigma^2)}$$

(where $r^2 = x^2 + y^2 + z^2$) has a charge density FT given by

$$\tilde{\rho}(k) = (2\pi)^{3/2}\sigma^3\epsilon_0\, k^2 e^{-k^2\sigma^2/2},$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$.

and so the potential is sourced by a charge density field

$$\rho(\boldsymbol{r}) = \frac{\epsilon_0}{\sigma^2}\left(3 - \frac{r^2}{\sigma^2}\right) e^{-r^2/(2\sigma^2)}.$$

You may assume that the Fourier Transform (w.r.t. x, $\to k_x$) of $e^{-x^2/(2\sigma^2)}$ is $\sqrt{2\pi}\sigma\, e^{-k_x^2\sigma^2/2}$, and that $\int_{-\infty}^{\infty} e^{-u^2/2}du = \sqrt{2\pi}$. You can also assume the inverse FT of $k^2e^{-k^2\sigma^2/2}$ which you proved in the above question. [Hint: you will be faced with an integral with 3 terms in it (involving $k_x^2 + k_y^2 + k_z^2$). Do one of them only, and engage brain to write down the answer for the other two without doing more algebra].