Fourier Analysis

Workshop 7: Green's Functions

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1. Letting $p = dy/dt$, and then using an integrating factor, show that the general solution to

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

is $y(t) = A + Be^{-t}$, where A and B are constants. Hence show that the causal Green's function for this problem is

$$G(t,T) = \begin{cases} 0 & t < T \\ 1 - e^{-(t-T)} & t > T \end{cases}$$

Give a physical interpretation of this behaviour.

2. Show that the Green's function for the range $x \geq 0$, satisfying

$$\frac{\partial^2 G(x,z)}{\partial x^2} + G(x,z) = \delta(x-z)$$

with boundary conditions $G(x,z) = \partial G(x,z)/\partial x = 0$ at $x = 0$ is

$$G(x,z) = \begin{cases} \cos z \sin x - \sin z \cos x & x > z \\ 0 & x < z \end{cases}$$

3. Consider the equation, valid for $t \geq 0$

$$\frac{d^2 f}{dt^2} + 5\frac{df}{dt} + 6f = e^{-t},$$

subject to boundary conditions $f = 0$, $df/dt = 0$ at $t = 0$. Find the Green's function $G(t, z)$, showing is is zero for $t < z$, and for $t > z$ it is

$$G(t, z) = e^{2z-2t} - e^{3z-3t}.$$

(You may find the complementary function (homogeneous solution) by using a suitable trial function).

Hence show that the solution to the equation is

$$f(t) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}.$$

4. (a) Show that the Green's function for the equation, valid for $t \geq 0$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = f(t),$$

with $y = 0$ and $dy/dt = 0$ at $t = 0$, is

$$G(t, T) = \begin{cases} 0 & t < T \\ 1 - e^{T-t} & t > T \end{cases}.$$

(b) Hence show that if $f(t) = Ae^{-2t}$, the solution is

$$y(t) = \frac{A}{2}\left(1 - 2e^{-t} + e^{-2t}\right).$$

5. The equation for a driven, damped harmonic oscillator is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + (1 + k^2)y = f(t)$$

(a) If the initial conditions are $y = 0$ and $dy/dt = 0$ at $t = 0$, show that the Green's function, valid for $t \geq 0$, is

$$G(t, T) = \begin{cases} A(T)e^{-t}\cos kt + B(T)e^{-t}\sin kt & 0 < t < T \\ C(T)e^{-t}\cos kt + D(T)e^{-t}\sin kt & t > T \end{cases}$$

(b) Show that $A = B = 0$ and so $G(t, T) = 0$ for $t < T$.

(c) By matching $G(t, T)$ at $t = T$, and requiring dG/dt to have a discontinuity of 1 there, show that, for $t > T$

$$G(t, T) = \frac{e^{T-t}}{k}\left(-\sin kT \cos kt + \cos kT \sin kt\right).$$

(d) Hence if $f(t) = e^{-t}$, find the solution for $y(t)$.