Fourier Analysis

Workshop 6: Fourier solution of differential equations

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1. (a) By expanding both sides as Fourier Sin Series, show that the solution to the equation

$$\frac{d^2y}{dx^2} + y = 2x$$

with boundary conditions $y(x=0) = 0$, $y(x=1) = 0$ is

$$y(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(1-n^2\pi^2)} \sin(n\pi x).$$

(b) Show that the r.m.s. value of $y(x)$ is

$$\sqrt{\langle y^2(x) \rangle} = \frac{4}{\pi} \sqrt{\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2(1-n^2\pi^2)^2}}$$

2. If the function $f(x)$ is periodic with period 2π and has a complex Fourier Series representation

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx}$$

then show that the solution of the differential equation

$$\frac{dy}{dx} + ay = f(x)$$

is

$$y(x) = \sum_{n=-\infty}^{\infty} \frac{f_n}{a+in} e^{inx}.$$

3. An LCR series circuit has a sinusoidal voltage $V_0 \sin \omega t$ imposed, so the current I obeys:

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + CI = \omega V_0 \cos \omega t.$$

(a) What is the fundamental period of the voltage?
(b) Write $I(t)$ as a Fourier Series,

$$I(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

and show that a_n and b_n satisfy

$$\begin{aligned} C\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big\{ a_n \left[-Ln^2\omega^2 \cos(n\omega t) - Rn\omega \sin(n\omega t) + C\cos(n\omega t)\right] & \quad + \\ b_n \left[-Ln^2\omega^2 \sin(n\omega t) + Rn\omega \cos(n\omega t) + C \sin(n\omega t)\right] \Big\} & \quad = \quad \omega V_0 \cos \omega t. \end{aligned}$$

(c) Hence show that only a_1 and b_1 survive, with amplitudes

$$a_1 = \frac{\omega V_0(-L\omega^2 + C)}{(C - L\omega^2)^2 + R^2\omega^2}; \qquad b_1 = \frac{\omega^2 V_0 R}{(C - L\omega^2)^2 + R^2\omega^2}.$$

4. A simple harmonic oscillator with natural frequency ω_0 and no damping is driven by a driving acceleration term $f(t) = \sin t + \sin 2t$.

(a) Write down the differential equation obeyed by the displacement $y(t)$.

(b) Compute the fundamental period of the driving terms on the right hand side, and hence T (where the solution is assumed periodic on $-T < t < T$).

(c) Assuming the solution is periodic with the same fundamental period as the driving term, find the resultant motion.

(d) Calculate the r.m.s. displacement of the oscillator.

5. (a) Show that the FT of $h(t) = e^{-a|t|}$, for $a > 0$ is $\tilde{h}(\omega) = 2a/(a^2 + \omega^2)$.

(b) A system obeys the differential equation $d^2z/dt^2 - \omega_0^2 z = f(t)$. Calculate $\tilde{z}(\omega)$ in terms of $\tilde{f}(\omega)$.

(c) By considering the form of $\tilde{z}(\omega)$, show using the convolution theorem that a solution of the equation is the convolution of $f(t)$ with some function $g(t)$.

(d) Using your answer to (a), find the function $g(t)$ and write down explicitly a solution to the equation.