Fourier Analysis

Workshop 5: Convolutions, Correlations

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1. Write down the definition of a convolution of two functions, $f(x)$ and $g(x)$. By means of a change of variables, prove that $f * g = g * f$. If $g(x) = \delta(x - a)$, what is $f * g$?

2. Show that convolving a signal $f(t)$ with a Gaussian smoothing function

$$g(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

results in the Fourier Transform being 'low-pass filtered' with a weight $\exp(-\sigma^2\omega^2/2)$.

3. Show that the FT of a product $h(x) = f(x)g(x)$ is a convolution in k-space:

$$\tilde{h}(k) = \frac{1}{2\pi}\tilde{f}(k) * \tilde{g}(k) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \tilde{f}(k')\tilde{g}(k - k').$$

4. Show that the convolution of a Gaussian of width σ_1 with a Gaussian of width σ_2 gives another Gaussian, and calculate its width. (A Gaussian of width σ has the form $N \exp[-x^2/(2\sigma^2)]$).

5. The cross-correlation $h(x)$ of $f(x)$ and $g(x)$ is $h(x) = \int_{-\infty}^{\infty} f^*(x')g(x' + x)dx'$. Show that its Fourier transform is $\tilde{h}(k) = \tilde{f}^*(k)\tilde{g}(k)$, and hence prove the Wiener-Khinchin theorem relating the autocorrelation function and the power spectrum of a function $f(x)$.

6. A signal $f(x) = e^{-x}$ for $x > 0$ and is zero otherwise.

(a) Show that the Fourier Transform is $\tilde{f}(k) = (1 + ik)^{-1}$.

(b) Using Parseval's theorem, relate the integral of the power $|\tilde{f}(k)|^2$ to an integral of $|f(x)|^2$.

(c) Hence show that $\int_{-\infty}^{\infty} dk/(1+k^2) = \pi$.

(d) The signal is passed through a low-pass filter, which sets the Fourier coefficients to zero above $|k| = k_0$; calculate k_0 such that the filtered signal has 90% of the original power.

7. (a) Compute the Fourier Transform of

$$h(t) = \begin{cases} e^{-bt} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

(b) A system obeys the differential equation $dz/dt + \omega_0 z = f(t)$. By using the fact that the Fourier transform of dz/dt is $i\omega\tilde{z}(\omega)$,, show that a solution of the equation is a convolution of $f(t)$ with

$$g(t) = \begin{cases} e^{-\omega_0 t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Hence argue that the solution in this case can be written as

$$z(t) = \int_{-\infty}^{t} f(t') \exp[-\omega_0(t-t')]\, dt'.$$

8. Compute the Fourier Transform of

$$h(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that the convolution $H(x) \equiv h(x) * \delta(x-a)$ is 1 if $a-1 < x < a+1$ and zero otherwise, and compute its Fourier transform directly, and via the convolution theorem.

9. A triple slit experiment consists of slits that each have a Gaussian transmission with Gaussian width σ, and that are separated by a distance $d \gg \sigma$. Compute the intensity distribution far from the slits, and sketch it.