Fourier Analysis

Workshop 4: Dirac Delta Functions

Professor John A. Peacock
School of Physics and Astronomy
jap@roe.ac.uk
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1. By treating a delta-function as the limit of a top-hat function whose width tends to zero, prove the sifting property of the delta-function:

$$\int_{-\infty}^{\infty} f(q)\,\delta(x-q)\,dq = f(x).$$

What condition on f is required in order for this to be true? Show that $\int_a^b f(q)\,\delta(x-q)\,dq$ is zero if $x > b$ or $x < a$ (assuming $b > a$).

2. By using the result that if, for all functions $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\,g(x)\,dx = \int_{-\infty}^{\infty} f(x)\,h(x)\,dx$$

then $g(x) = h(x)$, show that

(a) $\delta(-x) = \delta(x)$
Hint: show that

$$\int_{-\infty}^{\infty} f(x)\delta(-x)dx = f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

(b) $\delta(ax) = \frac{\delta(x)}{|a|}$
(c) $\delta(x^2 - a^2) = \frac{\delta(x-a)+\delta(x+a)}{2|a|}$.
(d) $x\delta(x) = 0$.

3. Evaluate
(a) $\int_{-\infty}^{\infty} f(x)\,\delta(2x-3)\,dx$
(b) $\int_1^2 f(x)\,\delta(x-3)\,dx$

4. Evaluate

$$\int_{0.1}^{\pi+0.1} dx \int_{-1}^{4} dy\, \delta(\sin x)\delta(x^2 - y^2)$$

5. Show that the derivative of the Dirac delta function has the property that

$$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} f(t)\, dt = -\left.\frac{df}{dt}\right|_{t=0}$$

6. What are the Fourier Transforms of:
(a) $\delta(x)$
(b) $\delta(x - d)$
(c) $\delta(2x)$?
By writing $\delta(x)$ as an integral (i.e. as an Inverse Fourier Transform) show that
(d) $\delta^*(x) = \delta(x)$

7. (a) Find the Fourier transform of $h(x) \equiv H(x - a)\exp(-bx)$, where H is the Heaviside function.

(b) [harder] Write the real and imaginary parts of $\tilde{h}$ separately for $a = 0$. Hence take the limit $b \to 0$ to show that the Fourier transform of $H(x)$ is $\pi\delta(k) - i/k$.

(c) Using this result, what is the Fourier transform of $dH(x)/dx$? Check your result using the properties of the delta-function.

(d) What is the Fourier transform of $H(-x)$?

8. Evaluate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t^2 e^{-iax}\, e^{itx}\, dx\, dt$$

where a is a constant.

9. Compute

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixy}\, e^{-ixz}\, e^{ibz}\, e^{iyt}\, e^{-t^3}\, dx\, dy\, dz\, dt$$

where b is a constant.