Fourier Analysis

Workshop 3: Fourier Transforms

Professor John A. Peacock
School of Physics and Astronomy
jap@roe.ac.uk
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1. Prove that, for a real function $f(x)$, its Fourier Transform satisfies $\tilde{f}(-k) = \tilde{f}^*(k)$.

2. If $\tilde{f}(k)$ is the Fourier transform of $f(x)$, show that the Fourier transform of df/dx is $ik\tilde{f}$.

3. In terms of $\tilde{f}(k)$, the Fourier Transform of $f(x)$, what are the Fourier Transforms of the following?
(a) $g(x) = f(-x)$
(b) $g(x) = f(2x)$
(c) $g(x) = f(x+a)$
(d) $g(x) = df/dx$.
(e) $g(x) = xf(x)$

4. Consider a Gaussian quantum mechanical wavefunction

$$\psi(x) = A \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where A is a normalisation constant, and the width of the Gaussian is σ.
(a) Compute the Fourier Transform $\tilde{\psi}(k)$ and show that it is also a Gaussian.
(b) Noting that the probability density function is $|\psi(x)^2|$, show by inspection that the uncertainty in x (by which we mean the width of the Gaussian) is $\Delta x = \sigma/\sqrt{2}$.
(c) Then, using de Broglie's relation between the wavenumber k and the momentum, $p = \hbar k$, compute the uncertainty in p, and demonstrate Heisenberg's Uncertainty Principle,

$$\Delta p \Delta x = \frac{\hbar}{2}.$$

5. Express the Fourier Transform of $g(x) \equiv e^{iax} f(x)$ in terms of the FT of $f(x)$.

6. The function $f(x)$ is defined by

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$

(a) Calculate the FT of $f(x)$, and, using Q5, of $e^{ix} f(x)$ and of $e^{-ix} f(x)$.

(b) Hence show that the FT of $g(x) = f(x) \sin x$ is

$$\frac{1}{(1+ik)^2 + 1}$$

(c) Finally, calculate the FT of $h(x) = f(x) \cos x$.

7. (a) Show that the FT of $f(x) = e^{-a|x|}$ is $\tilde{f}(k) = 2a/(a^2 + k^2)$, if $a > 0$.

(b) Sketch the FT of the cases $a = 1$ and $a = 3$ on the same graph, and comment on the widths.

(c) Using the result of question 5, show that the FT of $g(x) = e^{-|x|} \sin x$ is

$$\tilde{g}(k) = \frac{-4ik}{4 + k^4}.$$

8. Let

$$h_a(x) \equiv \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

(a) Show that the FT of $h_a(x)$ is

$$\tilde{h}_a(k) = \frac{1}{a + ik}.$$

(b) Take the FT of the equation

$$\frac{df}{dx}(x) + 2f(x) = h_1(x)$$

and show that

$$\tilde{f}(k) = \frac{1}{1+ik} - \frac{1}{2+ik}.$$

(c) Hence show that $f(x) = e^{-x} - e^{-2x}$ is a solution to the equation (for $x > 0$).

(d) Verify your answer by solving the equation using an integrating factor.

(e) Comment on any difference in the solutions.

9. Compute the Fourier Transform of a top-hat function of height h and width $2a$, which is centred at $x = d = a$. Sketch the real and imaginary parts of the FT.