Fourier Analysis

Workshop 2: More on Fourier Series & Parseval

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Session: 2014/15
30th September & 3rd October 2014

1. (a) Show that the Fourier Series for $f(x) = x$ in the range $-\pi < x < \pi$ is

$$f(x) = 2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx.$$

(b) Hence, by carefully choosing a value of x, show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4}.$$

2. Compute the *complex* Fourier Series for $f(x) = x$, $-\pi < x < \pi$.

3. Compute the complex Fourier Series for

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ +1 & 0 < x < 1 \end{cases}$$

and show it is $c_n = (\cos(k_n) - 1)i/k_n$.

4. Using a trigonometric identity, or otherwise, compute the Fourier Series for $f(x) = x \sin x$ for $-\pi < x < \pi$, and hence show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\times 3} - \frac{1}{3\times 5} + \frac{1}{5\times 7} - \ldots$$

5. Prove Parseval's theorem for the complex Fourier Series. Write down the relation between the complex Fourier coefficients, c_m and the (a_m, b_m) coefficients for a real series, and hence deduce Parseval's theorem for a sin + cos series:

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2\,dx = |a_0/2|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right).$$

6. If $f(x) = |x|$ for $-\pi \leq x \leq \pi$,
(a) show that its Fourier Series is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos[(2n+1)x]}{(2n+1)^2}.$$

(b) Hence show that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

(c) State Parseval's theorem, and prove that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

7. You are *given* that the Fourier Series of $f(x) = x$ (defined for $-1 \leq x \leq 1$) is

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n\pi}\sin(n\pi x).$$

Using Parseval's theorem, show that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$