**FOURIER ANALYSIS: LECTURE 15**

# 9 Green's functions

## 9.1 Response to an impulse

We have spent some time so far in applying Fourier methods to solution of differential equations such as the damped oscillator. These equations are all in the form of

$$Ly(t) = f(t), \tag{9.169}$$

where $L$ is a *linear* differential operator. For the damped harmonic oscillator, $L = (d^2/dt^2 + \gamma d/dt + \omega_0^2)$. As we know, linearity is an important property because it allows superposition: $L(y_1 + y_2) = Ly_1 + Ly_2$. It is this property that lets us solve equations in general by the method of *particular integral plus complementary function*: guess a solution that works for the given driving term on the RHS, and then add any solution of the homogeneous equation $Ly = 0$; this is just adding zero to each side, so the sum of the old and new $y$ functions still solves the original equation.

In this part of the course, we focus on a very powerful technique for finding the solution to such problems by considering a very simple form for the RHS: an impulse, where the force is concentrated at a particular instant. A good example would be striking a bell with a hammer: the subsequent ringing is the solution to the equation of motion. This *impulse response function* is also called a *Green's function* after George Green, who invented it in 1828 (note the apostrophe: this is not a Green function). We have to specify the time at which we apply the impulse, $T$, so the applied force is a delta-function centred at that time, and the Green's function solves

$$LG(t, T) = \delta(t - T). \tag{9.170}$$

Notice that the Green's function is a function of $t$ and of $T$ separately, although in simple cases it is also just a function of $t - T$.

This may sound like a peculiar thing to do, but the Green's function is everywhere in physics. An example where we can use it without realising is in electrostatics, where the electrostatic potential satisfies Poisson's equation:

$$\nabla^2\phi = -\rho/\epsilon_0, \tag{9.171}$$

where $\rho$ is the charge density. What is the Green's function of this equation? It is the potential due to a unit charge at position vector $\boldsymbol{q}$ (leaving aside the factor $\epsilon_0$):

$$G(\boldsymbol{r}, \boldsymbol{q}) = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{q}|}. \tag{9.172}$$

## 9.2 Superimposing impulses

The reason it is so useful to know the Green's function is that a general RHS can be thought of as a superposition of impulses, just as a general charge density arises from summing individual point charges. We have seen this viewpoint before in interpreting the sifting property of the delta-function. To use this approach here, take $LG(t, T) = \delta(t - T)$ and multiply both sides by $f(T)$ (which is a constant). But now integrate both sides over $T$, noting that $L$ can be taken outside the integral because it doesn't depend on $T$:

$$L \int G(t, T) f(T)\, dT = \int \delta(t - T) f(T)\, dT = f(t). \tag{9.173}$$

The last step uses sifting to show that indeed adding up a set of impulses on the RHS, centred at differing values of $T$, has given us $f(t)$. Therefore, the general solution is a superposition of the different Green's functions:

$$y(t) = \int G(t, T) f(T)\, dT. \tag{9.174}$$

This says that we apply a force $f(T)$ at time $T$, and the Green's function tells us how to propagate its effect to some other time $t$ (so the Green's function is also known as a *propagator*).

### 9.2.1 Importance of boundary conditions

When solving differential equations, the solution is not unique until we have applied some boundary conditions. This means that the Green's function that solves $LG(t, T) = \delta(t - T)$ also depends on the boundary conditions. This shows the importance of having boundary conditions that are *homogeneous*: in the form of some linear constraint(s) being zero, such as $y(a) = y(b) = 0$, or $y(a) = \dot{y}(b) = 0$. If such conditions apply to $G(t, T)$, then a solution that superimposes $G(t, T)$ for different values of $T$ will still satisfy the boundary condition. This would not be so for $y(a) = y(b) = 1$, and the problem would have to be manipulated into one for which the boundary conditions were homogeneous – by writing a differential equation for $z \equiv y - 1$ in that case.

## 9.3 Finding the Green's function

The above method is general, but to find the Green's function it is easier to restrict the form of the differential equation. To emphasise that the method is not restricted to dependence on time, now consider a spatial second-order differential equation of the general form

$$\frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y(x) = f(x). \tag{9.175}$$

Now, if we can solve for the *complementary function* (i.e. solve the equation for zero RHS), the Green's function can be obtained immediately. This is because a delta function vanishes almost everywhere. So if we now put $f(x) \to \delta(x - z)$, then the solution we seek is a solution of the homogeneous equation everywhere except at $x = z$.

We split the range into two, $x < z$, and $x > z$. In each part, the r.h.s. is zero, so we need to solve the homogeneous equation, subject to the boundary conditions at the edges. At $x = z$, we have to be careful to match the solutions together. The $\delta$ function is infinite here, which tells us that the *first* derivative must be discontinuous, so when we take the second derivative, it diverges. The first derivative must change discontinuously by 1. To see this, integrate the equation between $z - \epsilon$ and $z + \epsilon$, and let $\epsilon \to 0$:

$$\int_{z-\epsilon}^{z+\epsilon} \frac{d^2 y}{dx^2} dx + \int_{z-\epsilon}^{z+\epsilon} a_1(x) \frac{dy}{dx} dx + \int_{z-\epsilon}^{z+\epsilon} a_0(x) dx = \int_{z-\epsilon}^{z+\epsilon} \delta(x - z) dx. \tag{9.176}$$

The second and third terms vanish as $\epsilon \to 0$, as the integrands are finite, and the r.h.s. integrates to 1, so

$$\left.\frac{dy}{dx}\right|_{z+\epsilon} - \left.\frac{dy}{dx}\right|_{z-\epsilon} = 1. \tag{9.177}$$

Note that the boundary conditions are important. If $y = 0$ on the boundaries, then we can add up the Green's function solutions with the appropriate weight. If the Green's function is zero on the boundary, then any integral of $G$ will also be zero on the boundary and satisfy the conditions.

### 9.3.1 Example

Consider the differential equation

$$\frac{d^2y}{dx^2} + y = x \tag{9.178}$$

with boundary conditions $y(0) = y(\pi/2) = 0$.

The Green's function is continuous at $x = z$, has a discontinuous derivative there, and satisfies the same boundary conditions as $y$. From the properties of the Dirac delta function, except at $x = z$, the Green's function satisfies

$$\frac{d^2G(x,z)}{dx^2} + G(x,z) = 0. \tag{9.179}$$

(Strictly, we might want to make this a partial derivative, at fixed $z$. It is written this way so it looks like the equation for $y$). This is a harmonic equation, with solution

$$G(x,z) = \begin{cases} A(z)\sin x + B(z)\cos x & x < z \\ C(z)\sin x + D(z)\cos x & x > z. \end{cases} \tag{9.180}$$

We now have to adjust the four unknowns $A, B, C, D$ to match the boundary conditions.

The boundary condition $y = 0$ at $x = 0$ means that $B(z) = 0$, and $y = 0$ at $x = \pi/2$ implies that $C(z) = 0$. Hence

$$G(x,z) = \begin{cases} A(z)\sin x & x < z \\ D(z)\cos x & x > z. \end{cases} \tag{9.181}$$

Continuity of $G$ implies that $A(z)\sin z = D(z)\cos z$ and a discontinuity of 1 in the derivative implies that $-D(z)\sin z - A(z)\cos z = 1$. We have 2 equations in two unknowns, so we can eliminate $A$ or $D$:

$$-A(z)\frac{\sin^2 z}{\cos z} - A(z)\cos z = 1 \Rightarrow A(z) = \frac{-\cos z}{\sin^2 z + \cos^2 z} = -\cos z \tag{9.182}$$

and consequently $D(z) = -\sin z$. Hence the Green's function is

$$G(x,z) = \begin{cases} -\cos z \sin x & x < z \\ -\sin z \cos x & x > z \end{cases} \tag{9.183}$$

The solution for a driving term $x$ on the r.h.s. is therefore (be careful here with which solution for $G$ to use: the first integral on the r.h.s. has $x > z$)

$$y(x) = \int_0^{\pi/2} z\, G(x,z)\, dz = -\cos x \int_0^x z \sin z\, dz - \sin x \int_x^{\pi/2} z \cos z\, dz. \tag{9.184}$$

Integrating by parts,

$$y(x) = (x\cos x - \sin x)\cos x - \frac{1}{2}(\pi - 2\cos x - 2x\sin x)\sin x = x - \frac{\pi}{2}\sin x. \tag{9.185}$$

## 9.4 Summary

So to recap, the procedure is to find the Green's function by

- replacing the driving term by a Dirac delta function
- solving the homogeneous equation either side of the impulse, *with the same boundary conditions e.g. $G = 0$ at two boundaries, or $G = \partial G/\partial x = 0$ at one boundary.*
- Note the *form* of the solution will be the same for (e.g.) $x < z$ and $x > z$, but the *coefficients* (strictly, they are not constant coefficients, but rather functions of $z$) will differ either side of $x = z$).
- matching the solutions at $x = z$ (so $G(x, z)$ is continuous).
- introducing a discontinuity of 1 in the first derivative $\partial G(x, z)/\partial x$ at $x = z$
- integrating the Green's function with the actual driving term to get the full solution.

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## 9.5 Example with boundary conditions at the same place/time

A mouse wishes to steal a piece of cheese from an ice-rink at the winter olympics. The cheese, which has a mass of 1 kg, is conveniently sitting on a frictionless luge of negligible mass. The mouse attaches a massless string and pulls, starting at $t = 0$. Unfortunately the mouse gets tired very quickly, so the force exerted declines rapidly $f(t) = e^{-t}$ (SI units). Find, using Green's functions, the resulting motion of the cheese, $z(t)$ and its terminal speed.

The equation to be solved is

$$\frac{d^2z}{dt^2} = e^{-t}. \tag{9.186}$$

Since the cheese is 'sitting' on the luge, we take the boundary conditions to be

$$z = 0; \qquad \frac{dz}{dt} = 0 \quad \text{at } t = 0. \tag{9.187}$$

We can, of course, solve this equation very easily simply by integrating twice, and applying the boundary conditions. As an exercise, we are going to solve it with Green's functions. This also makes the point that there is often more than one way to solve a problem.

For an impulse at $T$, the Green's function satisfies

$$\frac{\partial G(t, T)}{\partial t^2} = \delta(t - T) \tag{9.188}$$

so for $t < T$ and $t > T$ the equation to be solved is $\partial^2 G/\partial t^2 = 0$, which has solution

$$G(t, T) = \begin{cases} A(T)t + B(T) & t < T \\ C(T)t + D(T) & t > T \end{cases} \tag{9.189}$$

Now, we apply the same boundary conditions. $G(t=0)=0 \Rightarrow B=0$. The derivative $G'(t=0)=0 \Rightarrow A=0$, so $G(t,T)=0$ for $t<T$. This makes sense when one thinks about it. We are applying an impulse at time $T$, so until the impulse is delivered, the cheese remains at rest.

Continuity of $G$ at $t=T$ implies

$$C(T)T + D(T) = 0, \tag{9.190}$$

and a discontinuity of 1 in the derivative at $T$ implies that

$$C(T) - A(T) = 1. \tag{9.191}$$

Hence $C=1$ and $D=-T$ and the Green's function is

$$G(t,T) = \begin{cases} 0 & t<T \\ t-T & t>T \end{cases} \tag{9.192}$$

The full solution is then

$$z(t) = \int_0^\infty G(t,T)f(T)dT \tag{9.193}$$

where $f(T)=e^{-T}$. Hence

$$z(t) = \int_0^t G(t,T)f(T)dT + \int_t^\infty G(t,T)f(T)dT. \tag{9.194}$$

The second integral vanishes, because $G=0$ for $t<T$, so

$$z(t) = \int_0^t (t-T)e^{-T}dT = t\left[-e^{-T}\right]_0^t - \left\{\left[-Te^{-T}\right]_0^t + \int_0^t e^{-T}dT\right\} \tag{9.195}$$

which gives the motion as

$$z(t) = t - 1 + e^{-t}. \tag{9.196}$$

We can check that $z(0)=0$, that $z'(0)=0$, and that $z''(t)=e^{-t}$. The final speed is $z'(t\to\infty)=1$, so the cheese moves at 1 $\mathrm{m\,s^{-1}}$ at late times. Note that this technique can solve for an arbitrary driving term, obtaining the solution as an integral. This can be very useful, even if the integral cannot be done analytically, as a numerical solution may still be useful.

## 9.6 Causality

The above examples showed how the boundary conditions influence the Green's function. If we are thinking about differential equations in time, there will often be a different boundary condition, which is set by causality. For example, write the first equation we considered in a form that emphasises that it is a harmonic oscillator:

$$\ddot{G}(t,T) + \omega_0^2 G(t,T) = \delta(t-T). \tag{9.197}$$

Since the system clearly cannot respond before it is hit, the boundary condition for such applications would be expected on physical grounds to be

$$G(t,T) = 0 \quad (t<T). \tag{9.198}$$

Whether or not such behaviour is achieved depends on the boundary conditions. Our first example did not satisfy this criterion, because the boundary conditions were of the form $y(a)=y(b)=0$.

This clearly presents a problem if $T$ is between the points $a$ and $b$: it's as if the system knows when we will strike the bell, or how hard, in order that the response as some future time $t = b$ will vanish. In contrast, our second example with boundary conditions at a single point ended up yielding causal behaviour automatically, without having to put it in by hand.

The causal Green's function is particularly easy to find, because we only need to think about the behaviour at $t > T$. Here, the solution of the homogeneous equation is $A \sin \omega_0 t + B \cos \omega_0 t$, which must vanish at $t = T$. Therefore it can be written as $G(t, T) = A \sin[\omega_0(t - T)]$. The derivative must be unity at $t = T$, so the causal Green's function for the undamped harmonic oscillator is

$$G(t, T) = \frac{1}{\omega_0} \sin[\omega_0(t - T)]. \qquad (9.199)$$

### 9.6.1 Comparison with direct Fourier solution

As a further example, we can revisit again the differential equation with the opposite sign from the oscillator:

$$\frac{d^2 z}{dt^2} - \omega_0^2 z = f(t). \qquad (9.200)$$

We solved this above by taking the Fourier transform of each side, to obtain

$$z(t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{f}(\omega)}{\omega_0^2 + \omega^2} e^{i\omega t} \, d\omega. \qquad (9.201)$$

We then showed that this is in the form of a convolution:

$$z(t) = -\frac{1}{2\omega_0} \int_{-\infty}^{\infty} f(T) e^{-\omega_0 |t-T|} \, dT. \qquad (9.202)$$

This looks rather similar to the solution in terms of the Green's function, so can we say that $G(t, T) = -\exp(-\omega_0|t-T|)/2\omega_0$? Direct differentiation gives $\dot{G} = \pm \exp(-\omega_0|t-T|)/2$, with the + sign for $t > T$ and the − sign for $t < T$, so it has the correct jump in derivative and hence satisfies the equation for the Green's function.

But this is a rather strange expression, since it is symmetric in time: a response at $t$ can precede $T$. The problem is that we have imposed no boundary conditions. If we insist on causality, then $G = 0$ for $t < T$ and $G = A \exp[\omega_0(t - T)] + B \exp[-\omega_0(t - T)]$ for $t > T$. Clearly $A = -B$, so $G = 2A \sinh[\omega_0(t - T)]$. This now looks similar to the harmonic oscillator, and a unit step in $\dot{G}$ at $t = T$ requires

$$G(t, T) = \frac{1}{\omega_0} \sinh[\omega_0(t - T)]. \qquad (9.203)$$

So the correct solution for this problem will be

$$z(t) = \frac{1}{\omega_0} \int_{-\infty}^{t} f(T) \sinh[-\omega_0(t - T)] \, dT. \qquad (9.204)$$

Note the changed upper limit in the integral: forces applied in the future cannot affect the solution at time $t$. We see that the response, $z(t)$, will diverge as to increases, which is physically reasonable: the system has homogeneous modes that either grow or decline exponentially with time. Special care with boundary conditions would be needed if we wanted to excite only the decaying solution – in other words, this system is unstable.