FOURIER ANALYSIS: LECTURE 6

2.11.1 Convergence of Fourier series

Fourier series (real or complex) are very good ways of approximating functions in a finite range, by which we mean that we can get a good approximation to the function by using only the first few modes (i.e. truncating the sum over n after some low value $n = N$).

This is how music compression works in MP3 players, or how digital images are compressed in JPEG form: we can get a good approximation to the true waveform by using only a limited number of modes, and so all the modes below a certain amplitude are simply ignored.

We saw a related example of this in our approximation to π using Eqn. (2.79) and Table 1.

Not examinable:
Mathematically, this translates as the Fourier components converging to zero i.e. $a_n, b_n \to 0$ as $n \to \infty$, provided $f(x)$ is bounded (i.e. has no divergences). But how quickly do the high order coefficients vanish? There are two common cases:

1. The function and its first $p-1$ derivatives $(f(x), f'(x), \ldots f^{(p-1)}(x))$ are continuous, but the p^{th} derivative $f^{(p)}(x)$ has discontinuities:

$$a_n, b_n \sim 1/n^{p+1} \quad \text{for large } n. \tag{2.82}$$

An example of this was our expansion of $f(x) = x^2$. When we periodically extend the function, there is a discontinuity in the gradient ($p = 1$ derivative) at the boundaries $x = \pm L$. We have already seen $a_n \sim 1/n^2$ as expected (with $b_n = 0$).

2. $f(x)$ is periodic and piecewise continuous (i.e. it has jump discontinuities, but only a finite number within one period):

$$\Rightarrow \quad a_n, b_n \sim 1/n \quad \text{for large } n. \tag{2.83}$$

An example of this is the expansion of the odd function $f(x) = x$, which jumps at the boundary. The Fourier components turn out to be $b_n \sim 1/n$ (with $a_n = 0$).

End of non-examinable section.

2.11.2 How close does it get? Convergence of Fourier expansions

We have seen that the Fourier components generally get smaller as the mode number n increases. If we truncate the Fourier series after N terms, we can define an error D_N that measures how much the truncated Fourier series differs from the original function: i.e. if

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]. \tag{2.84}$$

we define the error as

$$D_N = \int_{-L}^{L} dx \, |f(x) - f_N(x)|^2 \geq 0. \tag{2.85}$$

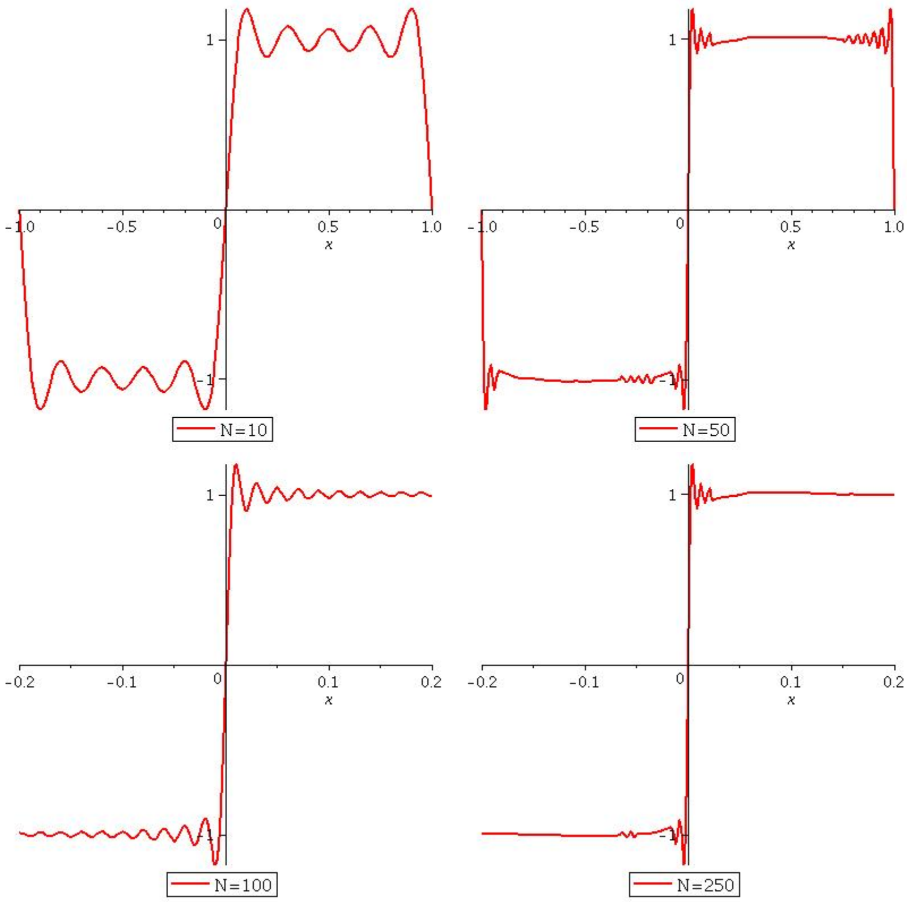

Figure 2.6: The Gibbs phenomenon for truncated Fourier approximations to the signum function Eqn. 2.86. Note the different x-range in the lower two panels.

That is, we square the difference between the original function and the truncated Fourier series at each point x, then integrate across the full range of validity of the Fourier series. Technically, this is what is known as an L^2 *norm.*

Some things you should know, but which we will not prove: if f is reasonably well-behaved (no non-integrable singularities, and only a finite number of discontinuities), the Fourier series is optimal in the least-squares sense – i.e. if we ask what Fourier coefficients will minimise D_N for some given N, they are exactly the coefficients that we obtain by solving the full Fourier problem.

Furthermore, as $N \to \infty$, $D_N \to 0$. This sounds like we are guaranteed that the Fourier series will represent the function exactly in the limit of infinitely many terms. But looking at the equation for D_N, it can be seen that this is not so: it's always possible to have (say) $f_N = 2f$ over some range Δx, and the best we can say is that Δx must tend to zero as N increases.

EXAMPLE: As an example of how Fourier series converge (or not), consider the signum function which picks out the sign of a variable:

$$f(x) = \operatorname{signum} x = \begin{cases} -1 & \text{if } x < 0 \,, \\ +1 & \text{if } x \geq 0 \,, \end{cases} \tag{2.86}$$

N	D_N
10	0.0808
50	0.0162
100	0.0061
250	0.0032

Table 2: Error D_N on the N-term truncated Fourier series approximation to the signum function Eqn. 2.86.

which we will expand in the range $-1 \leq x \leq 1$ (i.e. we set $L = 1$). The function is odd, so $a_n = 0$ and we find

$$b_n = 2 \int_0^1 dx\ \sin(n\pi x) = \frac{2}{n\pi} \left[1 - (-1)^n\right] \ . \tag{2.87}$$

$f(x)$ has discontinuities at $x = 0$ and $x = \pm L = \pm 1$ (due to the periodic extension), so from Sec. 2.11.1 we expected $a_n \sim 1/n$.

In Table 2 we show the error D_N for the signum function for increasing values of D_N. As expected the error decreases as N gets larger, but relatively slowly. We'll see why this is in the next section.

2.11.3 Ringing artefacts and the Gibbs phenomenon

We saw above that we can define an error associated with the use of a truncated Fourier series of N terms to describe a function. Note that D_N measures the total error by integrating the deviation at each value of x over the full range. It does not tell us whether the deviations between $f_N(x)$ and $f(x)$ were large and concentrated at certain values of x, or smaller and more evenly distributed over all the full range.

An interesting case is when we try to describe a function with a finite discontinuity (i.e. a jump) using a truncated Fourier series, such as our discussion of the signum function above.

In Fig. 2.6 we plot the original function $f(x)$ and the truncated Fourier series for various N. We find that the truncated sum works well, except near the discontinuity. Here the function overshoots the true value and then has a 'damped oscillation'. As we increase N the oscillating region gets smaller, but the overshoot remains roughly the same size (about 18%).

This overshoot is known as the *Gibbs phenomenon*. Looking at the plot, we can see that it tends to be associated with extended oscillations either side of the step, known as 'ringing artefacts'. Such artefacts will tend to exist whenever we try to describe sharp transitions with Fourier methods, and are one of the reasons that MP3s can sound bad when the compression uses too few modes. We can reduce the effect by using a smoother method of Fourier series summation, but this is well beyond this course. For the interested, there are some more details at `http://en.wikipedia.org/wiki/Gibbs_phenomenon`.

2.12 Parseval's theorem

There is a useful relationship between the mean square value of the function $f(x)$ and the Fourier coefficients. Parseval's formula is

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2\, dx = |a_0/2|^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) , \tag{2.88}$$

or, for complex Fourier Series,

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2\,dx = \sum_{n=-\infty}^{\infty}|c_n|^2. \tag{2.89}$$

The simplicity of the expression in the complex case is an example of the advantage of doing things this way.

The quantity $|c_n|^2$ is known as the *power spectrum*. This is by analogy with electrical circuits, where power is I^2R. So the mean of f^2 is like the average power, and $|c_n|^2$ shows how this is contributed by the different Fourier modes.

Proving Parseval is easier in the complex case, so we will stick to this. The equivalent for the sin+cos series is included for interest, but is not examinable. First, note that $|f(x)|^2 = f(x)f^*(x)$ and expand f and f^* as complex Fourier Series:

$$|f(x)|^2 = f(x)f^*(x) = \sum_{n=-\infty}^{\infty} c_n\phi_n(x)\sum_m c_m^*\phi_m^*(x) \tag{2.90}$$

(recall that $\phi_n(x) = e^{ik_nx}$). Then we integrate over $-L \le x \le L$, noting the orthogonality of ϕ_n and ϕ_m^*:

$$\begin{aligned}\int_{-L}^{L}|f(x)|^2\,dx &= \sum_{m,n=-\infty}^{\infty} c_nc_m^*\int_{-L}^{L}\phi_n(x)\phi_m^*(x)\,dx \qquad (2.91)\\ = \sum_{m,n=-\infty}^{\infty} c_nc_m^*(2L\delta_{mn}) &= 2L\sum_{n=-\infty}^{\infty}c_nc_n^* = 2L\sum_{n=-\infty}^{\infty}|c_n|^2\end{aligned}$$

where we have used the orthogonality relation $\int_{-L}^{L}\phi_n(x)\phi_m^*(x)\,dx = 2L$ if $m = n$, and zero otherwise.

2.12.1 Summing series via Parseval

Consider once again the case of $f = x^2$. The lhs of Parseval's theorem is $(1/2L)\int_{-L}^{L}x^4\,dx = (1/5)L^4$. The complex coefficients were derived earlier, so the sum on the rhs of Parseval's theorem is

$$\sum_{n=-\infty}^{\infty}|c_n|^2 = |c_0|^2 + \sum_{n\neq 0}|c_n|^2 = \left(\frac{L^2}{3}\right)^2 + 2\sum_{n=1}^{\infty}\left(\frac{2L^2(-1)^n}{n^2\pi^2}\right)^2 = \frac{L^4}{9} + \sum_{n=1}^{\infty}\frac{8L^4}{n^4\pi^4}. \tag{2.92}$$

Equating the two sides of the theorem, we therefore get

$$\sum_{n=1}^{\infty}\frac{1}{m^4} = (\pi^4/8)(1/5 - 1/9) = \pi^4/90. \tag{2.93}$$

This is a series that converges faster than the ones we obtained directly from the series at special values of x

FOURIER ANALYSIS: LECTURE 7

3 Fourier Transforms

Learning outcomes

In this section you will learn about Fourier transforms: their definition and relation to Fourier series; examples for simple functions; physical examples of their use including the diffraction and the solution of differential equations.

You will learn about the Dirac delta function and the convolution of functions.

3.1 Fourier transforms as a limit of Fourier series

We have seen that a Fourier series uses a complete set of modes to describe functions on a finite interval e.g. the shape of a string of length ℓ. In the notation we have used so far, $\ell = 2L$. In some ways, it is easier to work with ℓ, which we do below; but most textbooks traditionally cover Fourier series over the range $2L$, and these notes follow this trend.

Fourier transforms (FTs) are an extension of Fourier series that can be used to describe nonperiodic functions on an infinite interval. The key idea is to see that a non-periodic function can be viewed as a periodic one, but taking the limit of $\ell \to \infty$. This is related to our earlier idea of being able to construct a number of different periodic extensions of a given function. This is illustrated in Fig. 3.1 for the case of a square pulse that is only non-zero between $-a < x < +a$. When ℓ becomes large compared to a, the periodic replicas of the pulse are widely separated, and in the limit of $\ell \to \infty$ we have a single isolated pulse.

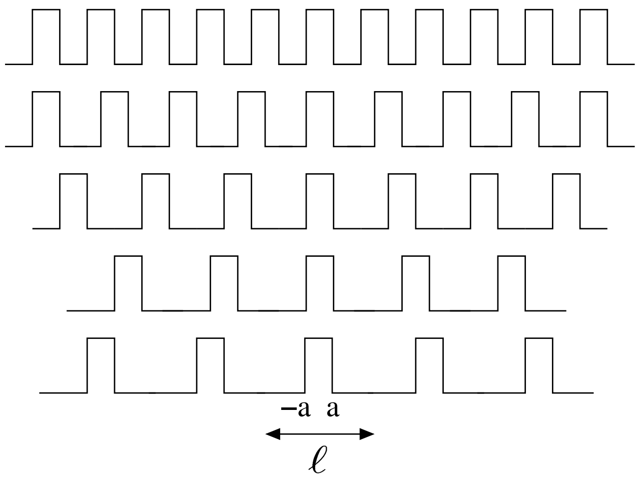

Figure 3.1: Different periodic extensions of a square pulse that is only non-zero between $-a < x < +a$. As the period of the extension, ℓ, increases, the copies of the pulse become more widely separated. In the limit of $\ell \to \infty$, we have a single isolated pulse and the Fourier series goes over to the Fourier transform.

Fourier series only include modes with wavenumbers $k_n = \frac{2n\pi}{\ell}$ with adjacent modes separated by $\delta k = \frac{2\pi}{\ell}$. What happens to our Fourier series if we let $\ell \to \infty$? Consider again the complex series for $f(x)$:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n x}, \tag{3.1}$$

where the coefficients are given by

$$C_n = \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} dx\ f(x)\ e^{-ik_n x}\ . \tag{3.2}$$

and the allowed wavenumbers are $k_n = 2n\pi/\ell$. The separation of adjacent wavenumbers (i.e. for $n \to n+1$) is $\delta k = 2\pi/\ell$; so as $\ell \to \infty$, the modes become more and more finely separated in k. In the limit, we are then interested in the variation of C as a function of the continuous variable k. The factor $1/\ell$ outside the integral looks problematic for talking the limit $\ell \to \infty$, but this can be evaded by defining a new quantity:

$$\tilde{f}(k) \equiv \ell \times C(k) = \int_{-\infty}^{\infty} dx\ f(x)\ e^{-ikx}\ . \tag{3.3}$$

The function $\tilde{f}(k)$ (officially called 'f tilde', but more commonly 'f twiddle'; f_k is another common notation) is the *Fourier transform* of the non-periodic function f.

To complete the story, we need the *inverse Fourier transform*: this gives us back the function $f(x)$ if we know $\tilde{f}$. Here, we just need to rewrite the Fourier series, remembering the mode spacing $\delta k = 2\pi/\ell$:

$$f(x) = \sum_{k_n} C(k) e^{ikx} = \sum_{k_n} (\ell/2\pi)\, C(k) e^{ikx}\ \delta k = \frac{1}{2\pi} \sum_{k_n} \tilde{f}(k)\, e^{ikx}\ \delta k. \tag{3.4}$$

In this limit, the final form of the sum becomes an integral over k:

$$\sum g(k)\, \delta k \to \int g(k)\, dk \quad \text{as} \quad \delta k \to 0; \tag{3.5}$$

this is how integration gets defined in the first place. We can now write an equation for $f(x)$ in which ℓ does not appear:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ \tilde{f}(k)\ e^{ikx}\ . \tag{3.6}$$

Note the infinite range of integration in k: this was already present in the Fourier series, where the mode number n had no limit.

EXAM TIP: You may be asked to explain how the FT is the limit of a Fourier Series (for perhaps 6 or 7 marks), so make sure you can reproduce the stuff in this section.

The density of states In the above, our sum was over individual Fourier modes. But if $C(k)$ is a continuous function of k, we may as well add modes in bunches over some bin in k, of size Δk:

$$f(x) = \sum_{k\ \text{bin}} C(k) e^{ikx}\, N_{\text{bin}}, \tag{3.7}$$

where $N_{\rm bin}$ is the number of modes in the bin. What is this? It is just Δk divided by the mode spacing, $2\pi/\ell$, so we have

$$f(x) = \frac{\ell}{2\pi} \sum_{k \text{ bin}} C(k) e^{ikx} \, \Delta k \tag{3.8}$$

The term $\ell/2\pi$ is the *density of states*: it tells us how many modes exist in unit range of k. This is a widely used concept in many areas of physics, especially in thermodynamics. Once again, we can take the limit of $\Delta k \to 0$ and obtain the integral for the inverse Fourier transform.

Summary A function $f(x)$ and its Fourier transform $\tilde{f}(k)$ are therefore related by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; \tilde{f}(k) \; e^{ikx} \; ; \tag{3.9}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \; f(x) \; e^{-ikx} \; . \tag{3.10}$$

We say that $\tilde{f}(k)$ is the FT of $f(x)$, and that $f(x)$ is the inverse FT of $\tilde{f}(k)$.

EXAM TIP: If you are asked to state the relation between a function and its Fourier transform (for maybe 3 or 4 marks), it is sufficient to quote these two equations. If the full derivation is required, the question will ask explicitly for it.

Note that, since the Fourier Transform is a *linear* operation,

$$FT[f(x) + g(x)] = \tilde{f}(k) + \tilde{g}(k). \tag{3.11}$$

For a real function $f(x)$, its FT satisfies the same Hermitian relation that we saw in the case of Fourier series:

$$\tilde{f}(-k) = \tilde{f}^*(k) \tag{3.12}$$

Exercise: prove this.

FT conventions Eqns. (3.10) and (3.9) are the definitions we will use for FTs throughout this course. Unfortunately, there are many different conventions in active use for FTs. Aside from using different symbols, these can differ in:

- The sign in the exponent
- The placing of the 2π prefactor(s) (and sometimes it is $\sqrt{2\pi}$)
- Whether there is a factor of 2π in the exponent

The bad news is that you will probably come across all of these different conventions. The good news is that that it is relatively easy to convert between them if you need to. The best news is that you will almost never need to do this conversion.

k space and momentum space The Fourier convention presented here is the natural one that emerges as the limit of the Fourier series. But it has the disadvantage that it treats the Fourier transform and the inverse Fourier transform differently by a factor of 2π, whereas in physics we need to learn to treat the functions $f(x)$ and $\tilde{f}(k)$ as equally valid forms of the same thing: the 'real-space' and 'k-space' forms. This is most obvious in quantum mechanics, where a wave function $\exp(ikx)$ represents a particle with a well-defined momentum, $p = \hbar k$ according to de Broglie's hypothesis. Thus the description of a function in terms of $\tilde{f}(k)$ is often called the 'momentum-space' version.

The result that illustrates this even-handed approach most clearly is to realise that the Fourier transform of $f(x)$ can itself be transformed:

$$\widetilde{\tilde{f}(k)}(K) = \int_{-\infty}^{\infty} dk\ \tilde{f}(k)\ e^{-iKk}. \tag{3.13}$$

We will show below that

$$\widetilde{\tilde{f}(k)}(K) = 2\pi f(-K)\ : \tag{3.14}$$

so in essence, repeating the Fourier transform gets you back the function you started with. f and $\tilde{f}$ are really just two sides of the same coin.

FOURIER ANALYSIS: LECTURE 8

3.2 Some simple examples of FTs

In this section we'll find the FTs of some simple functions.

EXAM TIP: You may be asked to define and sketch $f(x)$ in each case, and also to calculate and sketch $\tilde{f}(k)$.

3.2.1 The top-hat

A top-hat function $\Pi(x)$ of height h and width $2a$ (a assumed positive), centred at $x = d$ is defined by:

$$\Pi(x) = \begin{cases} h, & \text{if } d-a < x < d+a\ , \\ 0, & \text{otherwise}\ . \end{cases} \tag{3.15}$$

The function is sketched in Fig. 3.2.

Its FT is:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\ \Pi(x)\ e^{-ikx} = h\int_{d-a}^{d+a} dx\ e^{-ikx} = 2ah\ e^{-ikd}\ \mathrm{sinc}(ka) \tag{3.16}$$

The derivation is given below. The function $\mathrm{sinc}\,x \equiv \frac{\sin x}{x}$ is sketched in Fig. 3.3 (with notes on how to do this also given below). $\tilde{f}(k)$ will look the same (for $d = 0$), but the nodes will now be at $k = \pm\frac{n\pi}{a}$ and the intercept will be $2ah$ rather than 1. You are very unlikely to have to sketch $\tilde{f}(k)$ for $d \neq 0$.

EXAM TIPS: If the question sets $d = 0$, clearly there is no need to do a variable change from x to y.

Sometimes the question specifies that the top-hat should have unit area i.e. $h \times (2a) = 1$, so you can replace h.

The width of the top-hat won't necessarily be $2a$...

Deriving the FT:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, \Pi(x)\, e^{-ikx} = h \int_{d-a}^{d+a} dx\, e^{ikx} \tag{3.17}$$

Now we make a substitution $u = x - d$ (which now centres the top-hat at $u = 0$). The integrand e^{-ikx} becomes $e^{-ik(u+d)} = e^{-iku} \times e^{-ikd}$. We can pull the factor e^{-ikd} outside the integral because it does not depend on u. The integration limits become $u = \pm a$. There is no scale factor, i.e. $du = dx$.

This gives

$$\begin{aligned}\tilde{f}(k) &= he^{-ikd} \int_{-a}^{a} du\, e^{iku} = he^{-ikd} \left[\frac{e^{-iku}}{-ik}\right]_{-a}^{a} = he^{-ikd} \left(\frac{e^{-ika} - e^{ika}}{-ik}\right) \\ &= he^{-ikd} \times \frac{2a}{ka} \times \frac{e^{ika} - e^{-ika}}{2i} = 2ahe^{-ikd} \times \frac{\sin(ka)}{ka} = 2ah\, e^{-ikd}\ \mathrm{sinc}(ka)\end{aligned} \tag{3.18}$$

Note that we conveniently multiplied top and bottom by $2a$ midway through.

Sketching sinc x: You should think of $\mathrm{sinc}\, x \equiv \frac{\sin x}{x}$ as a $\sin x$ oscillation (with nodes at $x = \pm n\pi$ for integer n), but with the amplitude of the oscillations dying off as $1/x$. Note that $\mathrm{sinc}\, x$ is an even function, so it is symmetric when we reflect about the y-axis.

The only complication is at $x = 0$, when $\mathrm{sinc}\, 0 = \frac{0}{0}$ which appears undefined. To deal with this, expand $\sin x = x - x^3/3! + x^5/5! + \ldots$, so it is obvious that $\sin x/x \to 1$ as $x \to 0$.

EXAM TIP: Make sure you can sketch this, and that you label all the zeros ('nodes') and intercepts.

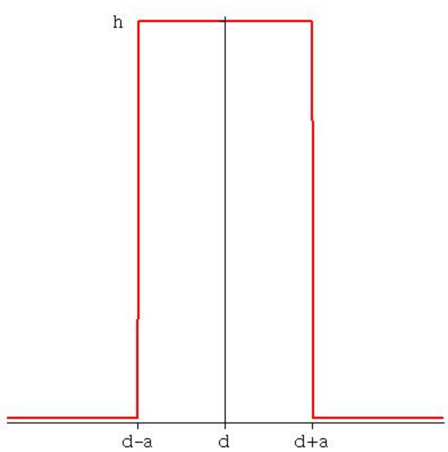

Figure 3.2: Sketch of top-hat function defined in Eqn. (3.15)

3.2.2 The Gaussian

The Gaussian curve is also known as the bell-shaped or normal curve. A Gaussian of width σ centred at $x = d$ is defined by:

$$f(x) = N \exp\left(-\frac{(x-d)^2}{2\sigma^2}\right) \tag{3.19}$$

where N is a normalization constant, which is often set to 1. We can instead define the *normalized* Gaussian, where we choose N so that the area under the curve to be unity i.e. $N = 1/\sqrt{2\pi\sigma^2}$. This normalization can be proved by a neat trick, which is to extend to a two-dimensional Gaussian for two independent (zero-mean) variables x and y, by multiplying the two independent Gaussian functions:

$$p(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}. \tag{3.20}$$

The integral over both variables can now be rewritten using polar coordinates:

$$\iint p(x,y)\, dx\, dy = \int p(x,y)\, 2\pi\, r\, dr = \frac{1}{2\pi\sigma^2} \int 2\pi\, r\, e^{-r^2/2\sigma^2}\, dr \tag{3.21}$$

and the final expression clearly integrates to

$$P(r > R) = \exp\left(-R^2/2\sigma^2\right), \tag{3.22}$$

so the distribution is indeed correctly normalized.

The Gaussian is sketched for $d = 0$ and two different values of the width parameter σ. Fig. 3.4 has $N = 1$ in each case, whereas Fig. 3.5 shows normalized curves. Note the difference, particularly in the intercepts.

For $d = 0$, the FT of the Gaussian is

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\; N \exp\left(-\frac{x^2}{2\sigma^2}\right)\; e^{-ikx} = \sqrt{2\pi} N\sigma \exp\left(-\frac{k^2\sigma^2}{2}\right), \tag{3.23}$$

i.e. the FT of a Gaussian is another Gaussian (this time as a function of k).

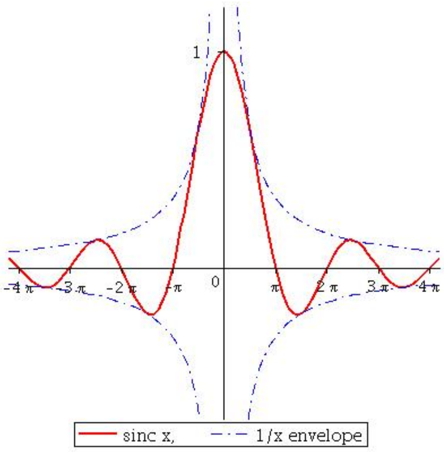

Figure 3.3: Sketch of $\operatorname{sinc} x \equiv \frac{\sin x}{x}$

Deriving the FT For notational convenience, let's write $a = \frac{1}{2\sigma^2}$, so

$$\tilde{f}(k) = N \int_{-\infty}^{\infty} dx \;\exp\left(-\left[ax^2 + ikx\right]\right) \tag{3.24}$$

Now we can complete the square inside [...]:

$$-ax^2 - ikx = -a\left(x + \frac{ik}{2a}\right)^2 - \frac{k^2}{4a} \tag{3.25}$$

giving

$$\tilde{f}(k) = Ne^{-k^2/4a} \int_{-\infty}^{\infty} dx \;\exp\left(-a\left[x + \frac{ik}{2a}\right]^2\right) . \tag{3.26}$$

We then make a change of variables:

$$u = \sqrt{a}\left(x + \frac{ik}{2a}\right) . \tag{3.27}$$

This does not change the limits on the integral, and the scale factor is $dx = du/\sqrt{a}$, giving

$$\tilde{f}(k) = \frac{N}{\sqrt{a}}\, e^{-k^2/4a} \int_{-\infty}^{\infty} du\, e^{-u^2} = N\sqrt{\frac{\pi}{a}} \times e^{-k^2/4a} = e^{-k^2/4a} . \tag{3.28}$$

where we changed back from a to σ. To get this result, we have used the standard result

$$\int_{-\infty}^{\infty} du\, e^{-u^2} = \sqrt{\pi} . \tag{3.29}$$

3.3 Reciprocal relations between a function and its FT

These examples illustrate a general *and very important* property of FTs: there is a reciprocal (i.e. inverse) relationship between the width of a function and the width of its Fourier transform. That is, narrow functions have wide FTs and wide functions have narrow FTs.

This important property goes by various names in various physical contexts, e.g.:

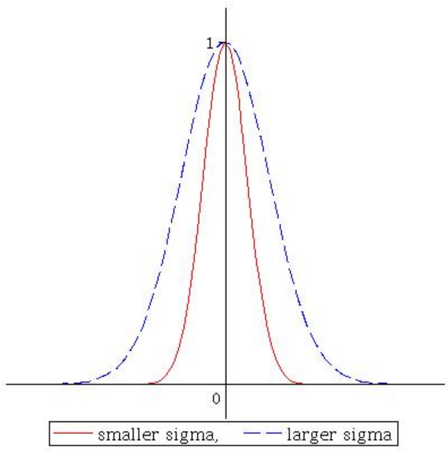

Figure 3.4: Sketch of Gaussians with $N = 1$

- Heisenberg Uncertainty Principle: the rms uncertainty in position space (Δx) and the rms uncertainty in momentum space (Δp) are inversely related: $(\Delta x)(\Delta p) \geq \hbar/2$. The equality holds for the Gaussian case (see below).
- Bandwidth theorem: to create a very short-lived pulse (small Δt), you need to include a very wide range of frequencies (large $\Delta\omega$).
- In optics, this means that big objects (big relative to wavelength of light) cast sharp shadows (narrow FT implies closely spaced maxima and minima in the interference fringes).

We discuss two explicit examples in the following subsections:

3.3.1 The top-hat

The width of the top-hat as defined in Eqn. (3.15) is obviously $2a$.

For the FT, whilst the $\mathrm{sinc}\, ka$ function extends across all k, it dies away in amplitude, so it does have a width. Exactly how we define the width does not matter; let's say it is the distance between the first nodes $k = \pm\pi/a$ in each direction, giving a width of $2\pi/a$.

Thus the width of the function is proportional to a, and the width of the FT is proportional to $1/a$. Note that this will be true for any reasonable definition of the width of the FT.

3.3.2 The Gaussian

Again, the Gaussian extends infinitely but dies away, so we can define a width. For a Gaussian, it is easy to do this rigorously in terms of the standard deviation (square root of the average of $(x - d)^2$), which is just σ (check you can prove this).

Comparing the form of FT in Eqn. (3.23) to the original definition of the Gaussian in Eqn. (3.19), if the width of $f(x)$ is σ, the width of $\tilde{f}(k)$ is $1/\sigma$ by the same definition. Again, we have a reciprocal relationship between the width of the function and that of its FT. Since $p = \hbar k$, the width in momentum space is $\hbar$ times that in k space.

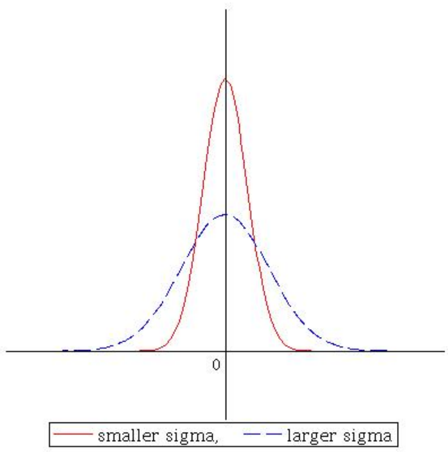

Figure 3.5: Sketch of normalized Gaussians. The intercepts are $f(0) = \frac{1}{\sqrt{2\pi\sigma^2}}$.

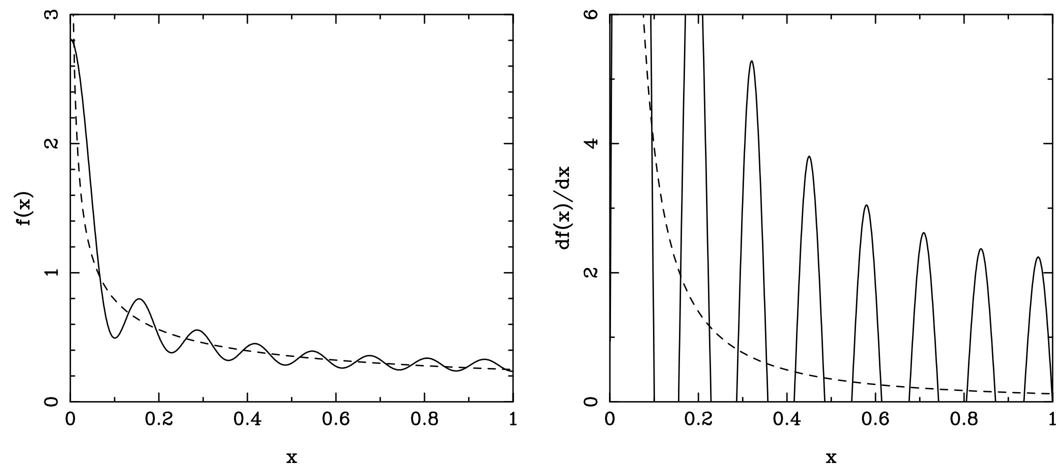

Figure 3.6: The Fourier expansion of the function $f(x) = 1/(4|x|^{1/2})$, $|x| < 1$ is shown in the LH panel (a cosine series, up to $n = 15$). The RH panel compares df/dx with the sum of the derivative of the Fourier series. The mild divergence in f means that the expansion converges; but for df/dx it does not.

The only subtlety in relating this to the uncertainty principle is that the probability distributions use $|\psi|^2$, not $|\psi|$. If the width of $\psi(x)$ is σ, then the width of $|\psi|^2$ is $\sigma/\sqrt{2}$. Similarly, the uncertainty in momentum is $(1/\sigma)/\sqrt{2}$, which gives the extra factor 1/2 in $(\Delta x)(\Delta p) = \hbar/2$.

3.4 Differentiating and integrating Fourier series

Once we have a function expressed as a Fourier series, this can be a useful alternative way of carrying out calculus-related operations. This is because differentiation and integration are linear operations that are *distributive* over addition: this means that we can carry out differentiation or integration term-by-term in the series:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n x} \tag{3.30}$$

$$\Rightarrow \frac{df}{dx} = \sum_{n=-\infty}^{\infty} C_n \, (ik_n) \, e^{ik_n x} \tag{3.31}$$

$$\Rightarrow \int f \; dx = \sum_{n=-\infty}^{\infty} C_n \, (ik_n)^{-1} \, e^{ik_n x} + \text{const} \; . \tag{3.32}$$

The only complication arises in the case of integration, if $C_0 \neq 0$: then the constant term integrates to be $\propto x$, and this needs to be handled separately.

From these relations, we can see immediately that the Fourier coefficients of a function and its derivative are very simply related by powers of k: if the n^{th} Fourier coefficient of $f(x)$ is C_n, the n^{th} Fourier coefficient of $df(x)/dx$ is $(ik_n)C_n$. Taking the limit of a non-periodic function, the same relation clearly applies to Fourier Transforms. Thus, in general, multiple derivatives transform as:

$$FT\left[f^{(p)}(x)\right] = FT\left[\frac{d^p f}{dx^p}\right] = (ik)^p \tilde{f}(k) \tag{3.33}$$

The main caveat with all this is that we still require that all the quantities being considered must be suitable for a Fourier representation, and this may not be so. For example, $f(x) = 1/\sqrt{x}$ for $0 < x < 1$ is an acceptable function: it has a singularity at $x = 0$, but this is integrable, so all the Fourier coefficients converge. But $f'(x) = -x^{-3/2}/2$, which has a divergent integral over $0 < x < 1$. Attempts to use a Fourier representation for $f'(x)$ would come adrift in this case, as is illustrated in Fig. 3.6.

FOURIER ANALYSIS: LECTURE 9

4 The Dirac delta function

The Dirac delta function is a very useful tool in physics, as it can be used to represent a very localised or instantaneous *impulse* whose effect then propagates. Informally, it is to be thought of as an infinitely narrow (and infinitely tall) spike. Mathematicians think it's not a proper function, since a function is a machine, $f(x)$, that takes any number x and replaces it with a well-defined number $f(x)$. Dirac didn't care, and used it anyway. Eventually, the 'theory of distributions' was invented to say he was right to follow his intuition.

4.1 Definition and basic properties

The Dirac delta function $\delta(x - d)$ is defined by *two* expressions. First, it is zero everywhere except at the point $x = d$ where it is infinite:

$$\delta(x-d) = \begin{cases} 0 & \text{for } x \neq d \ , \\ \to \infty & \text{for } x = d \ . \end{cases} \tag{4.34}$$

Secondly, it tends to infinity at $x = d$ in such a way that the area under the Dirac delta function is unity:

$$\int_{-\infty}^{\infty} dx \; \delta(x-d) = 1 \ . \tag{4.35}$$

4.1.1 The delta function as a limiting case

To see how a spike of zero width can have a well-defined area, it is helpful (although not strictly necessary) to think of the delta function as the limit of a more familiar function. The exact shape of this function doesn't matter, except that it should look more and more like a (normalized) spike as we make it narrower.

Two possibilities are the top-hat as the width $a \to 0$ (normalized so that $h = 1/(2a)$), or the normalized Gaussian as $\sigma \to 0$. We'll use the top-hat here, just because the integrals are easier.

Let $\Pi_a(x)$ be a normalized top-hat of width $2a$ centred at $x = 0$ as in Eqn. (3.15) — we've made the width parameter obvious by putting it as a subscript here. The Dirac delta function can then be defined as

$$\delta(x) = \lim_{a \to 0} \Pi_a(x) \ . \tag{4.36}$$

EXAM TIP: When asked to define the Dirac delta function, make sure you write *both* Eqns. (4.34) *and* (4.35).

4.1.2 Sifting property

The *sifting property* of the Dirac delta function is that, given some function $f(x)$:

$$\int_{-\infty}^{\infty} dx\ \delta(x-d)\ f(x) = f(d) \tag{4.37}$$

i.e. the delta function picks out the value of the function at the position of the spike (so long as it is within the integration range). This is just like the sifting property of the Kronecker delta inside a discrete sum.

EXAM TIP: If you are asked to state the sifting property, it is sufficient to write Eqn. (4.37). You do not need to prove the result as in Sec. 4.1.5 unless specifically asked to.

Technical aside: The integration limits don't technically need to be infinite in the above formulæ. If we integrate over a finite range $a < x < b$ the expressions become:

$$\int_{a}^{b} dx\ \delta(x-d) = \begin{cases} 1 & \text{for } a < d < b\ , \\ 0 & \text{otherwise.} \end{cases} \tag{4.38}$$

$$\int_{a}^{b} dx\ \delta(x-d)\ f(x) = \begin{cases} f(d) & \text{for } a < d < b\ , \\ 0 & \text{otherwise.} \end{cases} \tag{4.39}$$

That is, we get the above results if the position of the spike is inside the integration range, and zero otherwise.

4.1.3 Compare with the Kronecker delta

The Kronecker delta

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \tag{4.40}$$

plays a similar sifting role for discrete modes, as the Dirac delta does for continuous modes. For example:

$$\sum_{n=1}^{\infty} A_n \delta_{mn} = A_m \tag{4.41}$$

which is obvious when you look at it. Be prepared to do this whenever you see a sum with a Kronecker delta in it.

4.1.4 Delta function of a more complicated argument

Sometimes you may come across the Dirac delta function of a more complicated argument, $\delta[f(x)]$, e.g. $\delta(x^2 - 4)$. How do we deal with these? Essentially we use the definition that the delta function integrates to unity when it is integrated with respect to its argument. i.e.

$$\int_{-\infty}^{\infty} \delta[f(x)] df = 1 \tag{4.42}$$

Changing variables from f to x,

$$\int \delta[f(x)] \frac{df}{dx} dx = 1 \tag{4.43}$$

where we have not put the limits on x, as they depend on $f(x)$. Comparing with one of the properties of $\delta(x)$, we find that

$$\delta[f(x)] = \frac{\delta(x - x_0)}{|df/dx|_{x=x_0}} \tag{4.44}$$

where the derivative is evaluated at the point $x = x_0$ where $f(x_0) = 0$. Note that if there is more than one solution $(x_i; i = 1, \ldots)$ to $f = 0$, then $\delta(f)$ is a sum

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x=x_i}} \tag{4.45}$$

4.1.5 Proving the sifting property

We can use the limit definition of the Dirac delta function [Eqn. (4.36)] to prove the sifting property given in Eqn. (4.37):

$$\int_{-\infty}^{\infty} dx\ f(x)\ \delta(x) = \int_{-\infty}^{\infty} dx\ f(x) \lim_{a\to 0} \Pi_a(x) = \lim_{a\to 0} \int_{-\infty}^{\infty} dx\ f(x)\ \Pi_a(x)\ . \tag{4.46}$$

We are free to pull the limit outside the integral because nothing else depends on a. Substituting for $\Pi_a(x)$, the integral is only non-zero between $-a$ and a. Similarly, we can pull the normalization factor out to the front:

$$\int_{-\infty}^{\infty} dx\ f(x)\ \delta(x) = \lim_{a\to 0} \frac{1}{2a} \int_{-a}^{a} dx\ f(x)\ . \tag{4.47}$$

What this is doing is averaging f over a narrow range of width $2a$ around $x = 0$. Provided the function is *continuous*, this will converge to a well-defined value $f(0)$ as $a \to 0$ (this is pretty well the definition of continuity).

Alternatively, suppose the function was differentiable at $x = 0$ (which not all continuous functions will be: e.g. $f(x) = |x|$). Then we can Taylor expand the function around $x = 0$ (i.e. the position of the centre of the spike):

$$\int_{-\infty}^{\infty} dx\ f(x)\ \delta(x) = \lim_{a\to 0} \frac{1}{2a} \int_{-a}^{a} dx\ \left[f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots \right]\ . \tag{4.48}$$

The advantage of this is that all the $f^{(n)}(0)$ are constants, which makes the integral easy:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ f(x)\ \delta(x) &= \lim_{a\to 0} \frac{1}{2a} \left(f(0)\,[x]_{-a}^{a} + f'(0) \left[\frac{x^2}{2}\right]_{-a}^{a} + \frac{f''(0)}{2!} \left[\frac{x^3}{3}\right]_{-a}^{a} + \ldots \right) && (4.49)\\
&= \lim_{a\to 0} \left(f(0) + \frac{a^2}{6} f''(0) + \ldots \right) && (4.50)\\
&= f(0)\ . && (4.51)
\end{aligned}$$

which gives the sifting result.

EXAM TIP: An exam question may ask you to derive the sifting property in this way. Make sure you can do it.

Note that the odd terms vanished after integration. This is special to the case of the spike being centred at $x = 0$. It is a useful exercise to see what happens if the spike is centred at $x = d$ instead.

4.1.6 Some other properties of the delta function

These include:

$$\begin{aligned} \delta(-x) &= \delta(x) \\ x\delta(x) &= 0 \\ \delta(ax) &= \frac{\delta(x)}{|a|} \\ \delta(x^2 - a^2) &= \frac{\delta(x-a) + \delta(x+a)}{2|a|} \end{aligned} \tag{4.52}$$

The proofs are left as exercises.

4.1.7 Calculus with the delta function

The δ-function is easily integrated:

$$\int_{-\infty}^{x} dy \; \delta(y - d) = \Theta(x - d), \tag{4.53}$$

where

$$\Theta(x - d) = \begin{cases} 0 & x < d \\ 1 & x \geq d \end{cases} \tag{4.54}$$

which is called the *Heaviside function*, or just the 'step function'.

The derivative is also easy to write down, just by applying the product rule to $x\delta(x) = 0$:

$$\delta(x) + x\frac{d}{dx}\delta(x) = 0 \Rightarrow \frac{d}{dx}\delta(x) = -\delta(x)/x. \tag{4.55}$$

In the workshops, we will look at how the derivative of the δ-function can be used.

4.1.8 More than one dimension

Finally, in some situations (e.g. a point charge at $\boldsymbol{r} = \boldsymbol{r}_0$), we might need a 3D Dirac delta function, which we can write as a product of three 1D delta functions:

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \tag{4.56}$$

where $\boldsymbol{r}_0 = (x_0, y_0, z_0)$. Note that $\delta(\boldsymbol{r} - \boldsymbol{a})$ is not the same as $\delta(r - a)$: the former picks out a point at position $\boldsymbol{a}$, but the latter picks out an annulus of radius a. Suppose we had a spherically symmetric function $f(r)$. The sifting property of the 3D function is

$$\int f(\boldsymbol{r}) \; \delta(\boldsymbol{r} - \boldsymbol{a}) \; d^3x = f(\boldsymbol{a}) = f(a), \tag{4.57}$$

whereas

$$\int f(\boldsymbol{r}) \; \delta(r - a) \; d^3x = \int f(r) \; \delta(r - a) \; 4\pi r^2 \; dr = 4\pi a^2 f(a). \tag{4.58}$$

4.1.9 Physical importance of the delta function

The δ-function is a tool that arises a great deal in physics. There are a number of reasons for this. One is that the classical world is made up out of discrete particles, even though we often treat matter as having continuously varying properties such as density. Individual particles of zero size have infinite density, and so are perfectly suited to be described by δ-functions. We can therefore write the density field produced by a set of particles at positions $\boldsymbol{x}_i$ as

$$\rho(\boldsymbol{x}) = \sum_i M_i \delta(\boldsymbol{x} - \boldsymbol{x}_i). \tag{4.59}$$

This expression means we can treat all matter in terms of just the density as a function of position, whether the matter is continuous or made of particles.

This decomposition makes us look in a new way at the sifting theorem:

$$f(x) = \int f(q)\,\delta(x-q)\,dq. \tag{4.60}$$

The integral is the limit of a sum, so this actually says that the function $f(x)$ can be thought of as made up by adding together infinitely many δ-function spikes. This turns out to be an incredibly useful viewpoint when solving linear differential equations: the response of a given system to an applied force f can be calculated if we know how the system responds to a single spike. This response is called a *Green's function*, and will be a major topic later in the course.

4.2 FT and integral representation of $\delta(x)$

The Dirac delta function is very useful when we are doing FTs. The FT of the delta function follows easily from the sifting property:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\ \delta(x-d)\ e^{-ikx} = e^{-ikd}\ . \tag{4.61}$$

In the special case $d = 0$, we get simply $\tilde{f}(k) = 1$.

The inverse FT gives us the *integral representation* of the delta function:

$$\delta(x-d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ \tilde{f}(k)e^{ikx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{-ikd}e^{ikx} \tag{4.62}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ik(x-d)}\ . \tag{4.63}$$

You ought to worry that it's entirely unobvious whether this integral converges, since the integrand doesn't die off at ∞. A safer approach is to define the δ-function (say) in terms of a Gaussian of width σ, where we know that the FT and inverse FT are well defined. Then we can take the limit of $\sigma \to 0$.

In the same way that we have defined a delta function in x, we can also define a delta function in k. This would, for instance, represent a signal composed of oscillations of a single frequency or wavenumber K. Again, we can write it in integral form if we wish:

$$\delta(k-K) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-K)x}\,dx. \tag{4.64}$$

This k-space delta function has exactly the same sifting properties when we integrate over k as the original version did when integrating over x.

Note that the sign of the exponent is irrelevant:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm ikx}\, dk \tag{4.65}$$

which is easy to show by changing variable from k to $-k$ (the limits swap, which cancels the sign change $dk \to -dk$).

FOURIER ANALYSIS: LECTURE 10

5 Ordinary Differential Equations

We saw earlier that if we have a linear differential equation with a driving term on the right-hand side which is a periodic function, then Fourier Series may be a useful method to solve it. If the problem is not periodic, then Fourier Transforms may be able to solve it.

5.1 Solving Ordinary Differential Equations with Fourier transforms

The advantage of applying a FT to a differential equation is that we replace the *differential* equation with an *algebraic* equation, which may be easier to solve. Let us illustrate the method with a couple of examples.

5.1.1 Simple example

The equation to be solved is

$$\frac{d^2z}{dt^2} - \omega_0^2 z = f(t). \tag{5.66}$$

Take the FT, which for z is:

$$\tilde{z}(\omega) = \int_{-\infty}^{\infty} z(t) e^{-i\omega t}\, dt \tag{5.67}$$

and noting that $d/dt \to i\omega$, so $d^2/dt^2 \to -\omega^2$, the equation becomes

$$-\omega^2 \tilde{z}(\omega) - \omega_0^2 \tilde{z}(\omega) = \tilde{f}(\omega). \tag{5.68}$$

Rearranging,

$$\tilde{z}(\omega) = \frac{-\tilde{f}(\omega)}{\omega_0^2 + \omega^2} \tag{5.69}$$

with a solution

$$z(t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{f}(\omega)}{\omega_0^2 + \omega^2} e^{i\omega t}\, d\omega. \tag{5.70}$$

What this says is that a single oscillating $f(t)$, with amplitude a, will generate a response in phase with the applied oscillation, but of amplitude $a/(\omega_0^2 + \omega^2)$. For the general case, we superimpose oscillations of different frequency, which is what the inverse Fourier transform does for us.

Note that this gives a *particular* solution of the equation. Normally, we would argue that we can also add a solution of the homogeneous equation (where the rhs is set to zero), which in this case is $Ae^{\omega_0 t} + Be^{-\omega_0 t}$. Boundary conditions would determine what values the constants A and B take. But when dealing with Fourier transforms, this step may not be appropriate. This is because the Fourier transform describes non-periodic functions that stretch over an infinite range of time – so the manipulations needed to impose a particular boundary condition amount to a particular imposed force. If we believe that we have an expression for $f(t)$ that is valid for all times, then boundary conditions can only be set at $t = -\infty$. Physically, we would normally lack any reason for a displacement in this limit, so the homogeneous solution would tend to be ignored – even though it should be included as a matter of mathematical principle.

We will come back to this problem later, when we can go further with the calculation (see *Convolution*, section 6).

5.2 LCR circuits

Let us look at a more complicated problem with an electrical circuit. LCR circuits consist of an inductor of inductance L, a capacitor of capacitance C and a resistor of resistance R. If they are in series, then in the simplest case of one of each in the circuit, the voltage across all three is the sum of the voltages across each component. The voltage across R is IR, where I is the current; across the inductor it is LdI/dt, and across the capacitor it is Q/C, where Q is the charge on the capacitor:

$$V(t) = L\frac{dI}{dt} + RI + \frac{Q}{C}. \tag{5.71}$$

Now, since the rate of change of charge on the capacitor is simply the current, $dQ/dt = I$, we can differentiate this equation, to get a second-order ODE for I:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt}. \tag{5.72}$$

If we know how the applied voltage $V(t)$ varies with time, we can use Fourier methods to determine $I(t)$. With $\tilde{I}(\omega) = \int_{-\infty}^{\infty} I(t)e^{-i\omega t}\,dt$, and noting that the FT of dI/dt is $i\omega\tilde{I}(\omega)$, and of d^2I/dt^2 it is $-\omega^2\tilde{I}(\omega)$. Hence

$$-\omega^2 L\tilde{I}(\omega) + i\omega R\tilde{I}(\omega) + \frac{1}{C}\tilde{I}(\omega) = i\omega\tilde{V}(\omega), \tag{5.73}$$

where $\tilde{V}(\omega)$ is the FT of $V(t)$. Solving for $\tilde{I}(\omega)$:

$$\tilde{I}(\omega) = \frac{i\omega\tilde{V}(\omega)}{C^{-1} + i\omega R - \omega^2 L}, \tag{5.74}$$

and hence

$$I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\omega\tilde{V}(\omega)}{C^{-1} + i\omega R - \omega^2 L}\, e^{i\omega t}\,d\omega. \tag{5.75}$$

So we see that the individual Fourier components obey a form of Ohm's law, but involving a *complex impedance*, Z:

$$\tilde{V}(\omega) = Z(\omega)\tilde{I}(\omega); \quad Z = R + i\omega L - \frac{i}{\omega C}. \tag{5.76}$$

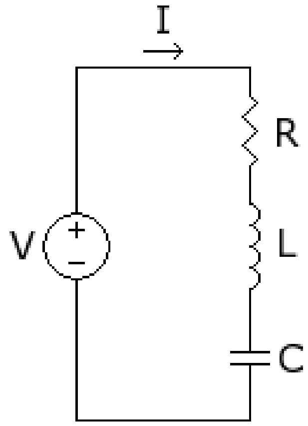

Figure 5.7: A simple series LCR circuit.

This is a very useful concept, as it immediately allows more complex circuits to be analysed, using the standard rules for adding resistances in series or in parallel.

The frequency dependence of the impedance means that different kinds of LCR circuit have functions as *filters* of the time-dependent current passing through them: different Fourier components (i.e. different frequencies) can be enhanced or suppressed. For example, consider a resistor and inductor in series:

$$\tilde{I}(\omega) = \frac{\tilde{V}(\omega)}{R + i\omega L}. \tag{5.77}$$

For high frequencies, the current tends to zero; for $\omega \ll R/L$, the output of the circuit (current over voltage) tends to the constant value $\tilde{I}(\omega)/\tilde{V}(\omega) = R$. So this would be called a *low-pass filter*: it only transmits low-frequency vibrations. Similarly, a resistor and capacitor in series gives

$$\tilde{I}(\omega) = \frac{\tilde{V}(\omega)}{R + (i\omega C)^{-1}}. \tag{5.78}$$

This acts as a *high-pass filter*, removing frequencies below about $(RC)^{-1}$. Note that the LR circuit can also act in this way if we measure the voltage across the inductor, V_L, rather than the current passing through it:

$$\tilde{V}_L(\omega) = i\omega L \tilde{I}(\omega) = i\omega L \frac{\tilde{V}(\omega)}{R + i\omega L} = \frac{\tilde{V}(\omega)}{1 + R(i\omega L)^{-1}}. \tag{5.79}$$

Finally, a full series LCR circuit is a *band-pass filter*, which removes frequencies below $(RC)^{-1}$ and above R/L from the current.

5.3 The forced damped Simple Harmonic Oscillator

The same mathematics arises in a completely different physical context: imagine we have a mass m attached to a spring with a spring constant k, and which is also immersed in a viscous fluid that exerts a resistive force proportional to the speed of the mass, with a constant of proportionality D. Imagine further that the mass is driven by an external force $f(t)$. The equation of motion for the displacement $z(t)$ is

$$m\ddot{z} = -kz - D\dot{z} + f(t). \tag{5.80}$$

This is the same equation as the LCR case, with

$$(z, f, m, k, D) \to (I, \dot{V}, L, C^{-1}, R). \tag{5.81}$$

To solve the equation for $z(t)$, first define a characteristic frequency by $\omega_0^2 = k/m$, and let $\gamma = D/m$. Then

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = a(t) \tag{5.82}$$

where $a(t) = f(t)/m$. Now take the FT of the equation with respect to time, and note that the FT of $\dot{z}(t)$ is $i\omega\tilde{z}(w)$, and the FT of $\ddot{z}(t)$ is $-\omega^2\tilde{z}(\omega)$. Thus

$$-\omega^2\tilde{z}(\omega) + i\omega\gamma\tilde{z}(\omega) + \omega_0^2\tilde{z}(\omega) = \tilde{a}(\omega), \tag{5.83}$$

where $\tilde{a}(\omega)$ is the FT of $a(t)$. Hence

$$\tilde{z}(\omega) = \frac{\tilde{a}(\omega)}{-\omega^2 + i\gamma\omega + \omega_0^2}. \tag{5.84}$$

5.3.1 Explicit solution via inverse FT

This solution in Fourier space is general and works for any time-dependent force. Once we have a specific form for the force, we can in principle use the Fourier expression to obtain the exact solution for $z(t)$. How useful this is in practice depends on how easy it is to do the integrals, first to transform $a(t)$ into $\tilde{a}(\omega)$, and then the inverse transform to turn $\tilde{z}(\omega)$ into $z(t)$. For now, we shall just illustrate the approach with a simple case.

Consider therefore a driving force that can be written as a single complex exponential:

$$a(t) = A\exp(\Omega t). \tag{5.85}$$

Fourier transforming, we get

$$\tilde{a}(\omega) = \int_{-\infty}^{\infty} Ae^{i\Omega t}\, e^{-i\omega t}\, dt = 2\pi A\delta(\Omega - \omega) = 2\pi A\delta(\omega - \Omega). \tag{5.86}$$

Unsurprisingly, the result is a δ-function spike at the driving frequency. Since we know that $\tilde{z}(\omega) = \tilde{a}(\omega)/(-\omega^2 + i\gamma\omega + \omega_0^2)$, we can now use the inverse FT to compute $z(t)$:

$$\begin{aligned} z(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{a}(\omega)}{-\omega^2 + i\gamma + \omega_0^2} e^{i\omega t}\, d\omega \\ &= A\int_{-\infty}^{\infty} \frac{\delta(\omega - \Omega)}{-\omega^2 + i\gamma\omega + \omega_0^2} e^{i\omega t}\, d\omega \\ &= A\,\frac{e^{i\Omega t}}{-\Omega^2 + i\gamma\Omega + \omega_0^2} \end{aligned} \tag{5.87}$$

This is just the answer we would have obtained if we had taken the usual route of trying a solution proportional to $\exp(i\Omega t)$ – but the nice thing is that the inverse FT has produced this for us automatically, without needing to guess.

5.3.2 Resonance

The result can be made a bit more intuitive by splitting the various factors into amplitudes and phases. Let $A = |A|\exp(i\phi)$ and $(-\Omega^2 + i\gamma\Omega + \omega_0^2) = a\exp(i\alpha)$, where

$$a = \sqrt{(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2} \tag{5.88}$$

and

$$\tan\alpha = \gamma\Omega/(\omega_0^2 - \Omega^2). \tag{5.89}$$

Then we have simply

$$z(t) = \frac{|A|}{a}\exp[i(\Omega t + \phi - \alpha)], \tag{5.90}$$

so the dynamical system returns the input oscillation, modified in amplitude by the factor $1/a$ and lagging in phase by α. For small frequencies, this phase lag is very small; it becomes $\pi/2$ when $\Omega = \omega_0$; for larger Ω, the phase lag tends to π.

The amplitude of the oscillation is maximized when a is a minimum. Differentiating, this is when we reach the *resonant frequency*:

$$\Omega = \Omega_{\rm res} = \sqrt{\omega_0^2 - \gamma^2/2}, \tag{5.91}$$

i.e. close to the natural frequency of the oscillator when γ is small. At this point, $a = (\gamma^2\omega_0^2 - \gamma^4/4)^{1/2}$, which is $\gamma\omega_0$ to leading order. From the structure of a, we can see that it changes by a large factor when the frequency moves from resonance by an amount of order γ (i.e. if γ is small then the width of the response is very narrow). To show this, argue that we want the term $(\omega_0^2 - \Omega^2)^2$, which is negligible at resonance, to be equal to $\gamma^2\omega_0^2$. Solving this gives

$$\Omega = (\omega_0^2 - \gamma\omega_0)^{1/2} = \omega_0(1 - \gamma/\omega_0)^{1/2} \simeq \omega_0 - \gamma/2. \tag{5.92}$$

Thus we are mostly interested in frequencies that are close to ω_0, and we can approximate a by

$$a \simeq [(\Omega^2 - \omega_0^2)^2 + \gamma^2\omega_0^2]^{1/2} \simeq [4\omega_0^2(\Omega - \omega_0)^2 + \gamma^2\omega_0^2]^{1/2}. \tag{5.93}$$

Thus, if we set $\Omega = \Omega_{\rm res} + \epsilon$, then

$$\frac{1}{a} \simeq \frac{(\gamma\omega_0)^{-1}}{(1 + 4\epsilon^2/\gamma^2)^{1/2}}, \tag{5.94}$$

which is a *Lorentzian* dependence of the square of the amplitude on frequency deviation from resonance.

5.3.3 Taking the real part?

The introduction of a complex acceleration may cause some concern. A common trick at an elementary level is to use complex exponentials to represent real oscillations; the argument being that (as long as we deal with linear equations) the real and imaginary parts process separately and so we can just take the real part at the end. Here, we have escaped the need to do this by saying that real functions require the Hermitian symmetry $c_{-m} = c_n^*$. If $a(t)$ is to be real, we must therefore also have the negative-frequency part:

$$\tilde{a}(\omega) = 2\pi A\delta(\omega - \Omega) + 2\pi A^*\delta(\omega + \Omega). \tag{5.95}$$

The Fourier transform of this is

$$a(t) = A\exp(i\Omega t) + A^* \exp(-i\Omega t) = 2|A|\cos(\Omega t + \phi), \tag{5.96}$$

where $A = |A|\exp(i\phi)$. Apart from a factor 2, this is indeed what we would have obtained via the traditional approach of just taking the real part of the complex oscillation.

Similarly, therefore, the time-dependent solution when we insist on this real driving force of given frequency comes simply from adding the previous solution to its complex conjugate:

$$z(t) = \frac{|A|}{a}\exp[i(\Omega t + \phi - \alpha)] + \frac{|A|}{a}\exp[-i(\Omega t + \phi - \alpha)] = 2\frac{|A|}{a}\cos(\Omega t + \phi - \alpha). \tag{5.97}$$