Fourier Analysis

Handin question 1

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1. A function that is defined over the interval $-L < x < L$ and which is assumed to be periodic outside this interval can be expressed as a complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp(ik_n x).$$

(a) What are the allowed values of the wavenumber, k_n? **[2]**

(b) Show that

$$a_m = \frac{1}{2L} \int_{-L}^{L} f(x) \exp(-ik_m x)\, dx.$$

[5]

(c) If $f(x)$ is an even function, show that the complex series reduces to a cosine series. Explain how the latter series can be written without using negative wavenumbers. **[5]**

(d) Consider the function $f(x) = x(\pi - x)$, defined over the range $0 < x < \pi$. Show that it can be written in both of the alternative forms

$$f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n^2} \quad \text{or} \quad \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin([2n-1]x)}{(2n-1)^3}.$$

Hint: $\int x^2 \cos(kx)\, dx$ and $\int x^2 \sin(kx)\, dx$ are given in the notes. **[10]**

(e) Hence show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

[3]