Fourier Analysis: December 2013

Section A: Answer all Questions

A.1 We wish to use a Fourier cosine series to calculate the function $f(x) = \exp(x)$ over the range $0 < x < 1$. Draw a sketch of the result of the Fourier series over the range $-2 < x < 2$; what is the fundamental period of this extended function? **[5]**

A.2 Prove that the Fourier transform, $\tilde{f}(k)$, of a real function, $f(x)$, obeys the symmetry $\tilde{f}(-k) = \tilde{f}^*(k)$. What is the corresponding relation for a purely imaginary function? **[5]**

A.3 Explain what is meant by the 'sifting property' of the Dirac delta-function, $\delta(x - a)$, where a is a constant. What is the Fourier transform of (a) $\delta(x)$? (b) $x^2\delta(x^2)$? **[5]**

A.4 A dynamical system has a response, $y(t)$, to a driving force, $f(t)$, that satisfies the differential equation $d^2y/dt^2 = f(t)$. Derive the causal Green's function for this system, $G(t, \tau)$, and explain how it can be used to solve the equation. **[5]**

Section B: Answer two Questions

B.1 A function $f(x)$ is given in the interval $-L < x < L$ and is to be expressed as a complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(ik_n x).$$

(a) Discuss how the function can be extended into a periodic form, and give the corresponding allowed values of the wavenumber, k_n. **[2]**

(b) Explain what is meant by the orthogonality of the Fourier modes $\exp(ik_n x)$. Assuming this property, derive an expression for the c_n coefficients in the form of an integral. **[4]**

(c) Hence show that a real function can also be written as a real series:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x),$$

defining the coefficients a_n and b_n, and explaining carefully why there is a factor 1/2 multiplying the a_0 term. **[4]**

(d) Consider the function $f(x) = x$, defined over the range $0 < x < \pi$. Show that it can be written in both of the following ways:

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 1]\frac{\cos(nx)}{n^2} \quad \text{or} \quad f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

[8]

(e) Hence show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

[2]

B.2 (a) Define the Fourier transform, $\tilde{f}(k)$, of a function $f(x)$, and give the inverse formula by which $f(x)$ can be obtained from its Fourier transform. **[3]**

(b) Compute the Fourier transform of a function that satisfies $f(x) = 1$ for $0 < x < a$ and is zero elsewhere. Discuss carefully the value of $\tilde{f}(0)$. **[4]**

(c) Prove that the Fourier transform of df/dx is $ik\tilde{f}(k)$. **[3]**

(d) Define the convolution, $f * g$, of two functions $f(x)$ and $g(x)$, and write down the relation that exists between the Fourier transforms of these functions, $\tilde{f}(k)$ and $\tilde{g}(k)$, and the transform of the convolution. What is the Fourier transform of $f(x)g(x)$? **[3]**

(e) Use Fourier methods to solve the following equation for $u(x,t)$, subject to the boundary condition $u(x,0) = \exp(-x^2/2)$. The Fourier transform of a Gaussian may be assumed without proof. **[5]**

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; \quad -\infty < x < \infty; \quad 0 < t < \infty.$$

(f) Interpret your result as a convolution, and hence give the solution for the case $u(x,0) = \delta(x-a)$. **[2]**

B.3 The equation of motion of a driven, damped harmonic oscillator is

$$\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = f(t).$$

(a) Consider the case where the driving term is an impulse at $t = 0$: $f(t) = \delta(t)$. Explain why the solution for this case, $y(t)$, must solve the homogeneous equation everywhere except at $t = 0$. If the oscillator is at rest with $y = 0$ prior to the impulse, describe how the solution changes in crossing from $t < 0$ to $t > 0$. **[3]**

(b) Hence show that, provided $\gamma < \omega_0$, the solution can be written as

$$y(t) = A\exp(-Bt)\sin(\Omega t) \quad (t > 0),$$

and give the values of the constants A, B and Ω. **[5]**

(c) How does the solution change when $\gamma > \omega_0$? **[2]**

(d) Hence write down the Green's function for this problem, where the driving force is $f(t) = \delta(t - T)$. **[3]**

(e) If $f(t) = \exp(-at)$ for $t > 0$, where $a > 0$, and is zero for $t < 0$, use the Green's function to find the resulting $y(t)$ for the case of $\gamma < \omega_0$. The indefinite integral $\int \exp(\alpha x)\sin x = \exp(\alpha x)(\alpha \sin x - \cos x)/(1 + \alpha^2)$ may be assumed. **[5]**

(f) If $a < \gamma$, show that the solution is dominated by a non-oscillatory term as $t \to \infty$. Prove that this limiting form is in fact an exact solution of the equation of motion. **[2]**

Section A: Solutions

A.1 For a cosine series, the function must be even. Therefore it is $\exp(-x)$ for $-1 < x < 0$. [BW] **[3]**

The simplest approach is to adopt a fundamental period of 2 and repeat the basic cell of $\exp(-|x|)$ over $-1 < x < 1$. But one could also repeat with a gap, so the fundamental period could be larger if we chose. [TQ] **[2]**

A.2 $\tilde{f} = \int f(x) \exp(-ikx)\, dx$ so $\tilde{f}^* = \int f(x) \exp(+ikx)\, dx$, which is $\tilde{f}(-k)$. [BW] **[3]**

If f was imaginary, we would have $\tilde{f}^*(k) = -\tilde{f}(-k)$. [US] **[2]**

A.3 Sifting means $\int f(x)\delta(x-a)\, dx = f(a)$. [BW] **[2]**

FT of $\delta(x)$ is 1. [BW] **[1]**

FT of $\delta(f(x))$ is $\delta(x-r)/|df/dx|_{x=r}$, summed over roots of the function, $f(r) = 0$. In this case, $r = 0$ only and $|df/dx|_{x=r} = 2|x|$. Hence $x^2\delta(x^2) = x^2\delta(x)/2|x| = |x|\delta(x)/2$. This means the FT integral is zero, since it sifts out $|x|/2$ at $x = 0$. [TQ] **[3]**

A.4 The Green's function must satisfy the homogeneous equation. Integrating directly, we get $G = a + b(t-\tau)$. Causality means $G = 0$ for $t < \tau$, so we need a unit step in G' at $t = \tau$, so G must vanish there. Thus $a = 0$ and $b = 1$. So $G(t, \tau) = (t-\tau)$. [TQ] **[3]**

G satisfies $d^2G/dt^2 = \delta(t-\tau)$. Multiply each side by $f(\tau)$ and integrate $d\tau$. The RHS becomes $f(t)$, so the solution to $y'' = f(t)$ is $\int_{-\infty}^{t} f(\tau)G(t,\tau)\, d\tau$ (upper limit in integral because $G = 0$ for $t < \tau$. [BW] **[2]**

Section B: Solutions

B.1 (a) We need to extend the function to a periodic form by replication. The simplest approach is to copy the 'cell' $-L < x < L$ into $L < x < 2L$ etc., with a fundamental period of $2L$. But the period could be larger: f is unknown outside the given range, so we could e.g. set $f = 0$ for $L < x < 2L$, and then copy that cell into $2L < x < 5L$, for a period of $\ell = 3L$. In any case, the wavenumber is $n(2\pi/\ell)$. [TQ] **[2]**

(b) $(1/\ell) \int \exp(ik_n x) \exp(-ik_m x)\, dx = \delta_{mn}$. [BW] **[2]**

Multiply the definition of f by $\exp(-ik_m x)$ and integrate, to obtain

$$c_m = (1/\ell) \int f(x) \exp(-ik_m x)\, dx.$$

[BW] **[2]**

(c) Split c_m into real and imaginary parts and allow for a factor 2 in usual definition of a_n and b_n. [TQ] **[2]**

For $n \neq 0$, we recover the factor 2 by combining the complex-series result for positive and negative n. [TQ] **[2]**

(d) Need to extend into $-\pi < x < 0$. Can choose $f(x) = x$ here (odd: sine series) or $f(x) = -x$ here (even: cosine series). [BW] **[3]**

The integrals needed are

$$a_m = \frac{2}{\pi}\int_0^\pi x\,\cos(nx)\,dx \quad \text{and} \quad b_m = \frac{2}{\pi}\int_0^\pi x\,\sin(nx)\,dx.$$

Doing these by parts gives $a_n = (2/\pi)(1/n^2)[\cos n\pi - 1]$ and $b_n = (2/\pi)(1/n)[0 - \pi\cos n\pi]$. And $\cos n\pi = (-1)^n$. a_0 can be derived directly or by expanding $\cos\epsilon \simeq 1 - \epsilon^2/2$. [TQ] **[5]**

(e) Set $x = 0$ in the cosine series. Hence $\pi^2/4 = \sum(2/n^2)$, where the sum is over odd n only. [TQ] **[2]**

B.2 (a) $\tilde{f}(k) = \int f(x)\exp(-ikx)\,dx$. $f(x) = (1/2\pi)\int \tilde{f}(k)\exp(ikx)\,dx$. [BW] **[3]**

(b) $\int_0^a \exp(-ikx)\,dx = (1/ik)[1 - \exp(-ika)]$. [US] **[2]**

This is $0/0$ for $k = 0$. Either Taylor expand for small k to get $ika/ik = a$ or do the integral with $k = 0$. [US] **[2]**

(c) Differentiate the integral: $f'(x) = (1/2\pi)\int \tilde{f}(k)ik\exp(ikx)\,dx$. [BW] **[1]**

Either by inspection, or by doing the FT directly and spotting a delta-function, the FT of f' is $ik\tilde{f}(k)$. [BW] **[2]**

(d) $f * g = \int f(x')g(x - x')\,dx'$. The FT of this is $\tilde{f}(k)\tilde{g}(k)$. [BW] **[2]**

The FT of $f(x)g(x)$ is $(1/2\pi)\tilde{f}(k) * \tilde{g}(k) = (1/2\pi)\int \tilde{f}(k') * \tilde{g}(k - k')'; dk'$. This can be quoted, or derived via a delta-function. [TQ] **[1]**

(e) Take the FT in space: $-k^2\tilde{u} = \frac{\partial}{\partial t}\tilde{u}$. This can be integrated directly to get $\tilde{u} = \tilde{u}(t = 0)\exp(-k^2 t)$. [TQ] **[2]**

Now FT $u(x, 0)$. This should be recognised as $\sqrt{2\pi}$ times a Gaussian with $\sigma = 1$, so $\tilde{u}(k, 0) = \sqrt{2\pi}\exp(-k^2/2)$. [BW] **[2]**

Hence $u(k, t) = \sqrt{2\pi}\exp(-k^2\sigma^2/2)$, where $\sigma^2 = 1 + 2t$, leading to

$$u(x, t) = (1/\sigma)\exp(-x^2/2\sigma^2).$$

[US] **[1]**

Because $\tilde{u}$ is a product, this represents convolving the initial $u(x, 0$ with a Gaussian of width $\sqrt{2}t$. Therefore convolving with a delta-function at a would give

$$u(x, t) = (\sqrt{4\pi}t)^{-1}\exp(-(x - a)^2/4t.$$

[US] **[2]**

B.3 (a) A delta-function is zero everywhere except at $t = 0$, so the equation is the homogeneous equation. Via causality, we need $y = 0$ for $t < 0$, and there must be a unit discontinuity in $\dot{y}$ at $t = 0$ to yield the delta-function. [BW] **[3]**

(b) Try $y = \exp(at)$, so $a^2 + 2\gamma a + \omega_0^2 = 0$, implying $a = -\gamma \pm i\Omega$, where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$. So for $\gamma < \omega_0$, we have

$$y = \exp(-\gamma t)[A \sin \Omega t + B \cos \Omega t].$$

We need $y(0) = 0$, so $B = 0$. Then

$$\dot{y} = A \exp(-\gamma t)[-\gamma \sin \Omega t + \Omega \cos \Omega t],$$

which is $A\Omega$ at $t = 0$. Thus a unit jump in $\dot{y}$ requires

$$y = \frac{1}{\Omega} \exp(-\gamma t) \sin \Omega t.$$

[TQ] **[5]**

(c) For $\gamma > \omega_0$, we have the same expression with $\sin \to \sinh$. [US] **[2]**

(d) Hence

$$G(t, T) = \frac{1}{\Omega} \exp[-\gamma(t - T)] \sin \Omega(t - T).$$

(obvious via a change of variables). [US] **[3]**

(e) $y(t) = \int_{-\infty}^{t} f(q)G(t, q)\, dq$. But $f(q) = 0$ for $q < 0$, so this is

$$y(t) = \frac{1}{\Omega} \int_0^t \exp(-aq) \exp[-\gamma(t - q)] \sin \Omega(t - q)\, dq.$$

Put $t - q = z$, giving

$$y(t) = \frac{1}{\Omega} \int_t^0 \exp(-at + az) \exp(-\gamma z) \sin \Omega z\, (-dz).$$

Now put $w = \Omega z$:

$$y(t) = \frac{\exp(-at)}{\Omega^2} \int_0^{\Omega t} \exp(aw/\Omega - \gamma w/\Omega) \sin w\, dw.$$

Using the supplied integral, this is

$$y(t) = [\Omega^2 + (a - \gamma)^2]^{-1} \left[\exp(-at) + \exp(-\gamma t) \left(\frac{a - \gamma}{\Omega} \sin \Omega t - \cos \Omega t\right)\right].$$

[US] **[5]**

(f) If $\gamma > a$ then the greater exponential damping of the second term means that it becomes negligible relative to the first at large t. In this limit,

$$y(t) = [\Omega^2 + (a - \gamma)^2]^{-1} \exp(-at).$$

Substituting this into the equation of motion gives a LHS of

$$(a^2 - 2\gamma a + \Omega^2 + \gamma^2)y = \exp(-at)$$

(using $\omega_0^2 = \Omega^2 + \gamma^2$). [US] **[2]**