MPhys Advanced Cosmology 2012/2013

Problem set 2

(1)

(a) Consider the universe at a GUT-scale temperature of $kT = 10^{15}$ GeV. Assuming a standard radiation-dominated model, estimate the comoving size of the particle horizon at the GUT era (currently, $kT = 10^{-3.6}$ eV).

(b) Write down the integral for the current size of the particle horizon. If the universe begins with a phase of inflation, during which the Hubble parameter, H, is constant, how long must inflation continue in order to reconcile the GUT-scale horizon with the present value of $\sim c/H_0$?

(c) Consider inflation driven by a single real scalar field, ϕ. Write down the exact equations for the time dependence of ϕ, and for the time dependence of the scale factor, a, in a universe containing only the inflaton.

(d) Explain how this equation of motion in the case of a homogeneous scalar field can be solved in the slow-roll approximation.

(e) For a potential $V(\phi) \propto \phi^\alpha$, show that the solution to the slow-roll equation is

$$\phi/\phi_{\rm initial} = (1 - t/t_{\rm final})^{1/\beta},$$

where $\beta = 2 - \alpha/2$. Explain why this equation does not hold near $\phi = 0$.

(f) Assuming curvature to be negligible, verify that a homogeneous scalar field with potential

$$V(\phi) \propto \exp\left(\sqrt{\frac{16\pi}{pm_{\rm P}^2}}\,\phi\right)$$

leads to inflation with a scale-factor dependence $a(t) \propto t^p$ (it helps to realise that this requires V and $\dot{\phi}^2$ to be in a fixed ratio, so that neither dominates).

(g) Evaluate the slow-roll parameters ϵ and η for this model, and show that the slow-roll approximation should be valid provided $p \gg 1$.

[Continued Overleaf]

(2) The horizon-scale amplitude predicted in single-field inflation models is $\delta_{\rm H} = H^2/(2\pi\dot{\phi})$.

(a) Show that, in the slow-roll approximation, this becomes

$$\delta_{\rm H}^2 = \frac{128\pi}{3}\left(\frac{V^3}{m_{\rm P}^6 V'^2}\right),$$

where V' stands for $dV/d\phi$.

(b) All quantities in this expression are assumed to be evaluated at horizon-crossing: $k/a = H$. Since H is nearly constant during inflation, the scale dependence of $\delta_{\rm H}^2$ can be evaluated by using

$$d/d\ln k = d/d\ln a = (\dot{\phi}/H)d/d\phi.$$

Hence prove the slow-roll expression for the tilt of the inflationary power spectrum:

$$\frac{d}{d\ln k}\ln\delta_{\rm H}^2 \equiv n - 1 = 2\eta - 6\epsilon.$$

(3)

(a) Consider an inflaton potential of the form of a double well:

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2.$$

Assuming an initial condition $\phi = \phi_i$ at $t = 0$, where ϕ_i is small and positive, show that the slow-roll equations have the solution

$$\phi(t) = \phi_i \exp(t/\tau),$$

and give an expression for the timescale τ.

(b) Show that inflation will not proceed unless the parameter v is $\gg m_p$, and hence that inflation is expected to end very near to $|\phi| = v$.

(c) Observed large-scale perturbations today exited the inflationary horizon $N \simeq 60$ e-foldings prior to the end of inflation. Derive an expression for N, and show that there are two possible situations, depending on whether or not v/m_p is large compared to $N/2\pi$.

(d) If v is large enough, show that observed scales also exited the horizon close to $\phi = v$. Since the potential is parabolic at this point, argue that the inflationary properties must be the same as for a $V = m^2\phi^2$ potential, i.e. $r = 4(1-n) = 16/(2N+1)$.

(e) If v is only modestly larger than m_p, show that observed scales exited the horizon at $\phi \ll v$, and that the predicted level of gravity waves is therefore low:

$$r = 16(1-n)e^{-[N(1-n)+1]}.$$