MPhys Advanced Cosmology 2012/2013

Problem set 1

(1) The Robertson-Walker metric can be written as

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\,\left(dr^2 + S_k^2(r)\,d\psi^2\right),$$

where $S_k(r) = \sin r$ $(k = +1)$, r $(k = 0)$, or $\sinh r$ $(k = -1)$ and $d\psi$ is the angular separation between the two events under consideration.

(a) Derive the radial equation of motion for a photon. Explain what is meant by the terms 'particle horizon' (= 'horizon', when used without qualification) and 'event horizon', and use the Friedmann equation to write these as integrals over redshift.

(b) Calculate the horizon size as a function of redshift for a flat universe containing matter and radiation; the integral is easier if you change variable from z to $a(z) = 1/(1+z)$. Show that the horizon size at matter-radiation equality is $(16.0/\Omega_m h^2)$ Mpc.

(c) Given that the current horizon radius in a flat vacuum-dominated model is approximately $(2c/H_0)\Omega_m^{-0.4}$, calculate the angle subtended today by the horizon at last scattering ($z = 1100$).

(d) Write down the integral for the event horizon in a flat universe containing matter and vacuum energy; argue that in general the integral converges. For pure de Sitter space, show that the event horizon with respect to $a = 0$ diverges in comoving length units, but that the event horizon with respect to events at a time t always takes the same value in proper length units.

(e) The North and South Hubble Deep Fields are two small patches that lie in opposite directions on the sky, and which contain statistically identical galaxy populations. Show that, in a flat vacuum-dominated cosmology, there are critical redshifts beyond which galaxies that we can observe in the two Hubble Deep Fields have not established causal contact (a) by the present day; (b) by the time at which the light we now see was emitted. Considering the following table of comoving distances for a flat $\Omega_m = 0.25$ model, estimate these redshifts.

z	$D(z)/\,h^{-1}\,\mathrm{Mpc}$
0.5	1345
1	2385
1.5	3178
2	3795
3	4690
5	5775
10	7051
∞	10666

(2)

(a) For the case of a flat universe containing only matter and a cosmological constant, show that the current age of the universe is

$$H_0 t_0 = \frac{2}{3}\,(1-\Omega_m)^{-1/2}\ \mathrm{arcsinh}\left[\left(\Omega_m^{-1}-1\right)^{1/2}\right]$$

(use the substitution $y = (1+z)^{-3/2}$).

(b) The expression in the previous part can be accurately approximated by $H_0 t_0 = (2/3)\Omega_m^{-0.3}$. By calculating the Hubble parameter and density parameter at non-zero redshift, show how this approximate expression can be extended to give the age of the universe at redshift z. Galaxies are known to exist at redshift 1.6 whose stellar populations are 3 Gyr old; what limit on H_0 would be required in order for this observation to provide evidence for vacuum energy, on the assumption that the universe is flat?

(3) The Friedmann equation can be written as

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2.$$

Show that in the limit $\Omega \to 1$, it is possible not only to neglect k in the Friedmann equation, but also to use the form of the Robertson-Walker metric in which the comoving part is uncurved (consider holding the local observables H and ρ fixed but increasing R without limit). Thus verify the connection between cosmological dynamics and geometry in the $k = 0$ case.

(4) From the definition $1 + z = 1/a(t)$, we can deduce $dz/dt = -(1+z)H(z)$. But here, t means look-back time: the change in cosmological time coordinate along a photon trajectory. But if we watch a distant object, its observed redshift will change as the expansion of the universe progresses. Show that

$$\frac{dz_{\rm obs}}{dt_{\rm obs}} = (1+z)H_0 - H(z).$$

For flat vacuum-dominated models, discuss the sign and magnitude of the effect. Apart from sheer instrumental precision, why might this be hard to detect?