Advanced Cosmology: Summer 2012

Section A: Answer two Questions

A.1

(a) The Robertson-Walker metric for an expanding universe can be written as

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)(dr^2 + S_k^2(r)d\psi^2).$$

Give the meaning of all terms, and write down the the Friedmann equation obeyed by the scale factor, $R(t)$. **[4]**

Show that, in the case of a closed universe, the spatial part of this metric is identical to that of the surface of a sphere. Hence, or otherwise, argue that radial coordinate $r = 2\pi$ corresponds to the same spatial point as $r = 0$. **[4]**

(b) Consider a universe containing only pressureless matter, with a density above critical. Show that such a universe will have a maximum scale factor, and hence recollapses to a big crunch. **[3]**

Using the radial equation of motion for light, calculate the particle horizon in such a universe at the point of maximum expansion, and hence prove that a photon that sets off at the big bang just returns to its starting point at the big crunch (you may assume that $\int_0^1 (x + x^2)^{-1/2} = \pi$). **[6]**

(c) Consider a universe containing pressureless matter and a cosmological constant. Show that such a model can be non-expanding provided it is closed. If the universe expands today with a given matter density parameter, use the two forms of the Friedmann equation to derive an equation for the vacuum density parameter that would yield a static non-expanding state in the infinite past. Verify that, if Ω_m is $\ll 1$, the solution is approximately $\Omega_v = 1 + 3(\Omega_m/2)^{2/3}$. Thus give an approximate expression for the maximum redshift that could be observed in such a universe, as a function of Ω_m. **[8]**

A.2

(a) Explain what is meant by freezeout in cosmology, and give a discussion of how the two main types of particle dark matter can arise in this way. How does the relic density of dark matter with a neutrino-like cross-section depend on mass, and what are the values of mass that yield the observed density? **[6]**

(b) A relic particle will have a proper peculiar velocity v, and the particle momentum is related to the energy, E, via $p = v(E/c^2)$, independent of whether or not the particle is relativistic. When the particle moves a proper distance δx, it meets an observer with velocity $\delta v = H\delta x$. Use a Lorentz transformation to show that the change in momentum is $\delta p/p = -\delta v(E/pc^2)$, and hence that p always scales $\propto 1/a(t)$, where $a(t)$ is the cosmic scale factor. **[6]**

(c) Hence give a rough estimate of the typical present-day peculiar velocity of a thermal relic particle that decouples when it is relativistic. Express your answer as a function of particle mass in eV. The typical energy of a CMB photon with $T = 2.725$ K is $kT = 0.000235$ eV. How is this velocity changed if decoupling occurs when the particle is non-relativistic? You may assume $mc^2/kT \simeq 10$ at decoupling in this case. **[6]**

(d) A collisionless relic particle moves at the speed of light, thus erasing all structure up to scales of the horizon, until it becomes non-relativistic. Show that the resulting free-streaming length, expressed in comoving units, is

$$L_{\mathrm{free-stream}} = 112\,(m/\mathrm{eV})^{-1}\,\mathrm{Mpc}.$$

Discuss the observational constraints that can be placed on the mass of a relic particle using this relation. You may assume the following relation between time and temperature while the universe is radiation-dominated: $t/\mathrm{s} = (T/10^{10.13}\mathrm{K})^{-2}$. **[7]**

A.3 The equation of motion for a homogeneous scalar field evolving under the action of a potential $V(\phi)$ is

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0,$$

and the Hubble parameter is given by

$$H^2 = \frac{8\pi}{3m_{\mathrm{P}}^2}(\dot{\phi}^2/2 + V).$$

(a) Explain how to solve these equations in the slow-roll limit. If the potential is of the mass-like form $V = m^2\phi^2/2$, what is the condition on the initial value of the field in order for this limit to apply? **[5]**

(b) Consider an inflaton potential of the form of a double well:

$$V(\phi) = \frac{1}{4\lambda}\,(\lambda\phi^2 - m^2)^2.$$

Assuming an initial condition $\phi = \phi_i$ at $t = 0$, where ϕ_i is small and positive, show that the slow-roll equations have the solution

$$\phi(t) = \phi_i \exp(t/\tau),$$

and give an expression for the timescale τ. **[6]**

(c) Explain why inflation must finish before $\dot{\phi}^2/2 = m^4/4\lambda$, and hence that it finishes close to the origin, at $\phi \simeq m^2\tau/\sqrt{2\lambda}$, provided $m/\sqrt{\lambda} \ll m_{\mathrm{P}}$. Thus show that a sufficiently long period of inflation requires

$$\phi_i < \frac{\tau m^2}{\sqrt{2\lambda}} \exp\left(-180\sqrt{\lambda}/\tau^2 m^2\right).$$

[9]

(d) Give a qualitative account of the history of the scalar field after inflation ends, and explain in outline how the transition to a radiation-dominated universe takes place. **[5]**