Advanced Cosmology: Summer 2009

Section A Answer two Questions

A.1.

(a) In a flat universe, the Hubble parameter as a function of epoch can be written as

$$H^2(a) = H_0^2 \left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v a^{-3(w+1)} \right).$$

Starting from the Friedmann equation, explain how this expression arises, and say how it is modified in non-flat models. **[5]**

(b) Write down the integrals for the relation between cosmological time and redshift, and for the relation between the particle horizon and redshift. Show that the proper size of the particle horizon during the early radiation-dominated era is $2ct$, and explain how this distance can be larger than ct. **[6]**

(c) For the case of a flat universe containing only matter and a cosmological constant, show that the current age of the universe is

$$H_0 t_0 = \frac{2}{3}\ (1 - \Omega_m)^{-1/2}\ \mathrm{arcsinh}\left[\left(\Omega_m^{-1} - 1 \right)^{1/2}\right]$$

(use the substitution $y = (1+z)^{-3/2}$). **[6]**

(d) The expression in the previous part can be accurately approximated by $H_0 t_0 = (2/3)\Omega_m^{-0.3}$. By calculating the Hubble parameter and density parameter at non-zero redshift, show how this approximate expression can be extended to give the age of the universe at redshift z. Galaxies are known to exist at redshift 1.6 whose stellar populations are 3 Gyr old; what limit on H_0 would be required in order for this observation to provide evidence for vacuum energy, on the assumption that the universe is flat? **[8]**

A.2. The equation of motion for a homogeneous scalar field evolving under the action of a potential $V(\phi)$ is

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0,$$

and the Hubble parameter is given by

$$H^2 = \frac{8\pi}{3m_{\mathrm{P}}^2}(\dot{\phi}^2/2 + V).$$

(a) Show that these equations are consistent with conservation of energy in the form

$$\frac{d\ln\rho}{d\ln a} = -3(1+w),$$

where w is the ratio between the pressure and energy density generated by the scalar field. **[6]**

(b) Suppose that the universe expands with $a(t) \propto t^p$ during inflation. Show that the potential required to produce this behaviour is

$$V(\phi) \propto \exp\left(\sqrt{\frac{16\pi}{p\,m_{\rm P}^2}}\;\phi\right)$$

(hint: assume that w is some constant, in order to express $\dot{\phi}$ in terms of V). **[7]**

(c) Explain how to solve the inflationary equations using the slow-roll approximation, and explain why this approximation requires that w be close to -1. What is the condition on the parameter p in the above power-law inflation model in order for the slow-roll condition to be obeyed? **[5]**

(d) Write notes on the end of inflation, explaining how this process differs in large-field and small-field models. **[7]**

A.3.

(a) The fractional density perturbation field, $\delta(\mathbf{x})$, is expressed as a Fourier integral;

$$\delta(\mathbf{x}) = \frac{V}{(2\pi)^3}\int \delta_k\;\exp(-i\mathbf{k}\cdot\mathbf{x})\;d^3k.$$

Explain the meaning of the terms in this expression and justify its form, explaining the relation to the alternative form where the density field is written as a discrete Fourier series. How would the Fourier coefficients, δ_k be obtained if the density field was known? **[4]**

(b) Explain how the expression in Fourier space makes it possible to find the gravitational potential that corresponds to a given density field. Discuss also the peculiar velocity field, which obeys the continuity equation $\nabla\cdot\mathbf{u} = -\partial\delta/\partial t$, and show how this may be solved if we assume $\mathbf{u}$ to be parallel to $\mathbf{k}$ in the growing mode. Explain the nature of a density disturbance with $\nabla\cdot\mathbf{u} = 0$, and give a physical reason why such modes to not grow with time. **[6]**

(c) If the density field has a power spectrum $|\delta_k|^2 = Ak^n$, give an expression for the variance in density that results when the density field is filtered by convolution with a uniform sphere of radius R (dimensionless integrals need not be evaluated exactly). For what values of the index n will the result be finite? How do the required limits on n change if we require a finite variance in (a) gravitational potential; (b) peculiar velocity? **[7]**

(d) A spherical region of space in a flat matter-dominated universe has a fractional density fluctuation of $\delta = 0.01$ at recombination ($z = 1100$). At some later redshift, it will have collapsed into a stable self-gravitating structure whose typical internal density is about 200 times the global mean density. Use the spherical collapse model to calculate this collapse redshift. By what factor will the internal density of the object exceed the global mean at the present day? **[8]**