MPhys Advanced Cosmology 2010/2011

Problem set 4

(1)

(a) When the universe cools to $T \ll 1000$ K, the material becomes primarily neutral. At late times, the remaining free electrons are removed at a rate determined simply by recombination. The rate at which electrons with proper number density n_e undergo recombination is approximately Rn_e, where $R \simeq 3 \times 10^{-17}(T/\mathrm{K})^{-1/2}\,\mathrm{m^3 s^{-1}}$. If $n_e = xn_e^{\mathrm{tot}}$, where x is the fractional ionization and

$$n_e^{\mathrm{tot}} = \Omega_b \,\frac{3H_0^2}{8\pi G \mu m_p}\,(1+z)^3,$$

where the parameter μ is approximately 1.143 for a gas of 25% helium by mass, discuss the freezeout of the ionization of the cosmic plasma. Show that the ionization does not fall below $x \simeq 10^{-4}$. At what redshift does this freezeout occur?

(b) Ionized material at low redshifts causes CMB photons to undergo Thomson scattering, which affects the observed CMB anisotropies. Show that the optical depth due to scattering between a redshift z and $z = 0$ is

$$\tau = \int \sigma_{\mathrm{T}}\, n_e\, d\ell_{\mathrm{prop}} = \int \sigma_{\mathrm{T}}\, n_e\, \frac{c}{H_0}\, \frac{dz}{(1+z)\sqrt{1-\Omega_m+\Omega_m(1+z)^3}}$$

(for a flat model).

(c) At low redshifts, structure formation causes reionization, so that $x = 1$ for $z < z_r$ Show that, for large z_r, the resulting optical depth is approximately

$$\tau = 0.04h\,\frac{\Omega_b}{\Omega_m}\left[\sqrt{1+\Omega_m z(3+3z+z^2)}-1\right] \simeq 0.04h\,\frac{\Omega_b}{\Omega_m^{1/2}}z^{3/2}.$$

(d) Describe the effect of this scattering on the pattern of CMB anisotropies.

(2) Give a simple proof that surface brightness is conserved in Euclidean space. Now show that volume elements in phase space are Lorentz invariant, and hence that the relativistic expression of surface-brightness conservation is that I_ν/ν^3 is an invariant. Show from this that black-body radiation appears thermal to all observers. If this is so, how is it possible to use the microwave background to determine that the Earth has an absolute velocity of $\simeq 370\ \mathrm{km\,s^{-1}}$?

(3)

(a) Show that the gravitational deflection of light can be described using a lensing potential ψ, which obeys a two-dimensional version of Poisson's equation.

(b) Derive the gravitational deflection formula $\alpha = 4GM(< r)/(c^2 r)$ for a mass distribution that is circular in projection, where M is the mass within a projected radius r (remember only the projected density matters).

(c) Show that the total flux amplification of a point source produced by a point-mass gravitational lens is

$$A = \frac{1 + x^2/2}{x\sqrt{1 + x^2/4}},$$

where $x \equiv \theta_{\mathrm{S}}/\theta_{\mathrm{E}}$, and derive an expression for the Einstein-ring radius θ_{E}. What are the amplifications of the individual images, and what shear would these images have, assuming that they can be resolved by a telescope?