MPhys Advanced Cosmology 2010/2011

Problem set 3

(1) The equation describing the growth of density fluctuations in a matter-dominated expanding universe is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \delta\big(4\pi G\bar{\rho}_m - c_s^2 k^2/a^2\big),$$

where δ is the fractional density fluctuation, $\rho = \bar{\rho}_m(1+\delta)$, $\bar{\rho}_m$ is the mean matter density, c_s is the speed of sound, and k is comoving wavenumber.

(a) Show that, for a static model, density fluctuations grow exponentially as long as the wavelength is sufficiently large, and explain physically why this is so.

(b) Derive the solutions to the perturbation equation for a universe of critical density, in the limit of infinitely long wavelength.

(c) At time $t = t_c$, a homogeneous critical-density universe is given a velocity perturbation, such that $\dot{\delta} = A$. Evaluate the density perturbation as a function of time following this event.

(d) If the universe contains a homogeneous component in addition to matter that can clump, the perturbation equation still applies – but $\bar{\rho}_m$ does not include the homogeneous component. Consider a flat universe that contains a mixture of hot and cold dark matter: for sufficiently small wavelengths, the hot component can be assumed to be uniform, with density parameter Ω_h. Show that fluctuations in the cold component grow as $\delta \propto t^\alpha$, where $\alpha = (\sqrt{25 - 24\Omega_h} - 1)/6$.

(2) At last scattering, $z = 1100$, an adiabatic density perturbation exists in the form of a uniform sphere of comoving radius $100\,h^{-1}$ Mpc, within which the density is 1.003 times the global mean. The universe is flat, with $\Omega_m = 0.25$.

(a) Discuss the mechanisms that can cause the CMB temperature from this direction on the sky to be perturbed.

(b) Show that only two of these mechanisms apply, and hence calculate the observed temperature perturbation, under the assumption that baryons and dark matter have the same density fluctuation. Why will this assumption be inexact?

(c) Show that there is a radius for the sphere at which the temperature perturbation would vanish, and calculate this critical radius.

(d) The sphere will increase in density contrast as the universe expands, and will eventually undergo gravitational collapse. Calculate the time at which this occurs, expressed as a redshift. Would this be the observed redshift of the object?

(e) If the sphere undergoes no further mergers after collapse, estimate its internal density contrast today.

(3) The photons that constitute the Cosmic Microwave background were last scattered at a mean redshift of 1070, with an rms dispersion in scattering redshift of 80. At this redshift, the relativistic density cannot be completely neglected. In a flat vacuum-dominated universe, a good approximation to the comoving distance to last scattering is $D_{\rm LS} = (2c/H_0)\Omega_m^{-0.4}$.

(a) Calculate the comoving thickness of the last-scattering shell. Assuming that this represents the smallest length-scale of surviving CMB temperature fluctuations, estimate the angular scale below which the CMB temperature would appear uniform.

(b) Derive an expression for the comoving horizon size at last scattering, and show that the angle that this length subtends today, $\theta_{\rm H-LS}$, is of order 1 degree. Explain without detailed calculation how this result would differ in an open universe with zero vacuum density.

(c) Explain in outline why this 1-degree scale is expected to be the dominant scale in the pattern of CMB temperature anisotropies.