MPhys Advanced Cosmology 2010/2011

Problem set 1

(1) The Robertson-Walker metric can be written as

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\,\left(dr^2 + S_k^2(r)\,d\psi^2\right),$$

where $S_k(r) = \sin r$ $(k = +1)$, r $(k = 0)$, or $\sinh r$ $(k = -1)$ and $d\psi$ is the angular separation between the two events under consideration.

(a) Derive the radial equation of motion for a photon. Explain what is meant by the terms 'particle horizon' (= 'horizon', when used without qualification) and 'event horizon', and use the Friedmann equation to write these as integrals over redshift.

(b) Calculate the horizon size as a function of redshift for a flat universe containing matter and radiation; the integral is easier if you change variable from z to $a(z) = 1/(1+z)$. Show that the horizon size at matter-radiation equality is $(16.0/\Omega_m h^2)$ Mpc.

(c) Given that the current horizon radius in a flat vacuum-dominated model is approximately $(2c/H_0)\Omega_m^{-0.4}$, calculate the angle subtended today by the horizon at last scattering ($z = 1100$).

(d) Write down the integral for the event horizon in a flat universe containing matter and vacuum energy; argue that in general the integral converges. For pure de Sitter space, show that the event horizon with respect to $a = 0$ diverges in comoving length units, but that the event horizon with respect to events at a time t always takes the same value in proper length units.

(2) The Friedmann equation can be written as

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2.$$

Show that in the limit $\Omega \to 1$, it is possible not only to neglect k in the Friedmann equation, but also to use the form of the Robertson-Walker metric in which the comoving part is uncurved (consider holding the local observables H and ρ fixed but increasing R without limit). Thus verify the connection between cosmological dynamics and geometry in the $k = 0$ case.

(3) From the definition $1+z=1/a(t)$, we can deduce $dz/dt=-(1+z)H(z)$. But here, t means look-back time: the change in cosmological time coordinate along a photon trajectory. But if we watch a distant object, its observed redshift will change as the expansion of the universe progresses. Show that

$$\frac{dz_{\rm obs}}{dt_{\rm obs}} = (1+z)H_0 - H(z).$$

For flat vacuum-dominated models, discuss the sign and magnitude of the effect. Apart from sheer instrumental precision, why might this be hard to detect?

(4) Show that adiabatic expansion of a perfect gas yields a dependence between temperature and scale factor of $T(a) \propto 1/a$ for a radiation-dominated gas, but $T(a) \propto 1/a^2$ for a nonrelativistic plasma. In practice, the temperature of the cosmic plasma follows that of the microwave background until a redshift about $z \sim 100$. Consider the heating of the plasma by Thomson scattering, and use the concept of freeze-out to explain why this is so.