First year PhD astrophysics reading group

Hand-in problems 2018/19 *Coordinator: Ken Rice*

(1) [Hand-in to John Peacock on Monday Oct 15]

A simplified galaxy model consists of a uniform slab of thickness L, containing stars mixed with dust. Assume that the dust has no scattering opacity, that the volume emissivity of starlight, $4\pi j_\nu$, is constant throughout the slab, and let τ_0 denote the optical depth for a light ray passing perpendicularly through the slab.

Calculate the surface brightness observed as a function of inclination. Show that it is a constant for large τ_0, but varies with angle in the optically thin limit. For the latter case, calculate the emergent flux density and verify that it satisfies conservation of energy.

Discuss qualitatively how these results would change if the dust is now given a mixture of scattering and absorption opacities. In the case of pure scattering, it can be assumed that the emergent radiation is isotropic. Use conservation of energy to calculate the surface brightness in this case.

(2) [Hand-in to Andy Lawrence on Monday Oct 22]

(a) An astronomer wishes to observe *Star A*, which has an R-band magnitude of 18.03 and is located at RA=$05^h32^m15.54^s$ and Dec=$+56°13'19.2''$. To locate this faint star, the astronomer decides to first centre their telescope on *Star B* at RA=$05^h32^m16.23^s$ and Dec=$+56°12'57.4''$, and then offset the telescope to *Star A*. How many arcseconds north/south and east/west does the telescope need to be offset? Sketch the relative positions of the two stars on the sky.

(b) A spectrum of *Star A* indicates that it is an A0 star, like Vega. What does this imply about the intrinsic optical colours of the star? Define what is meant by the term *Absolute magnitude*. Given that the absolute V-band magnitude of Vega is +0.6, determine the distance to *Star A*.

(c) At this magnitude, the Gaia satellite will be able to measure positions to an accuracy of 10^{-4} arcsec. Will it be able to measure the parallax of this star?

(d) The integrated flux of Vega is about $2.5 \times 10^{-8}\mathrm{W\,m^{-2}}$. What is the integrated flux of *Star A*? Hence, what is its luminosity in solar luminosities?

Assuming a surface temperature of 10,000 K, estimate the radius of *Star A* in solar units.

[$L_\odot = 3.85 \times 10^{26}$W, $R_\odot = 6.96 \times 10^8$m, $\sigma_{\rm SB} = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}K^{-4}}$]

(3) [Hand-in to Jim Dunlop on Monday Oct 29]

(a) A star of temperature T is surrounded by a cloud of neutral hydrogen. Explain how to calculate the rate at which the star emits photons capable of causing ionization.

(b) Show that the star creates an ionized nebula that grows in radius proportional to $t^{1/3}$ at early times, and explain the behaviour at late times in terms of radiative recombination.

(c) Calculate the kinetic energy given to an electron in a single ionization event, assuming that the stellar spectrum can be approximated as a power law $f_\nu \propto \nu^{-\alpha}$, and hence obtain a prediction for the temperature of the nebula if the energy loss rate is dominated by recombination.

(d) Explain how forbidden lines allow us to measure temperatures in nebulae, and also why these lines account for the fact that nebulae tend to be much cooler than their central stars.

(4) [Hand-in to Eric Tittley on Monday Nov 5]

For this problem set, you are going to extract regular signals from a noisy data set.

1. Download signal.dat (http://www.roe.ac.uk/∼ert/SignalProcessing/signal.dat).
2. Plot the signal, properly labelling the axes.
3. Compute the FFT of the signal.
4. Plot the FFT of the signal, with the axes properly labelled. The zero-th frequency should be at the centre of the figure. Pay special attention to the frequency labelling.
5. Determine the frequencies of any real signals. Report their frequencies and amplitudes (with errors). Errors should be one sigma. When determining the frequencies of the real signals, I would like 95% confidence (i.e. 95% confidence that there is a real signal with that frequency).
6. What does the spike at the zero-th frequency represent?

A note about the signal:

- One or more sinusoids on top of a noisy bias with a mean level of 1000 cts/s.
- All components of the signal are subject to Poisson noise.

There will be no perfect answer to this assignment. Just get as far as you can. I'm more interested in revealing to you things you don't know that you should, than determining for myself what you do. And not everything you need to know comes directly from the notes.

Use whatever computational tools you want. There is more than one way to do this project.

The writeup should be done in LaTeX(or TeX), though I won't necessarily be able to tell the difference if you do a good job in another document preparation system. I want publication-ready figures.

Books you may find useful for handling of error:

- Bevington & Robinson, "Data Reduction and Error Analysis for the Physical Sciences"
- Taylor, "An Introduction to Error Analysis"

(5) [Hand-in to Joe Zuntz on Monday Nov 12]

1. If you roll a 20-sided dice and cube the value shown, what is the expectation of the result?

2. The number of particles emitted by a radioactive source follows a poisson distribution. Element A emits on average 3.0 particles / minute / gram. Element B emits on average 4.0 particles / minute / gram. You have two boxes of element A and one box of element B. You have mixed up the three boxes at random, and taken out from one of them 1g of an element. You measure its radioactive emission for one minute and detect 5 particles.

   (a) What is the probability that the source is element A?

   (b) What is the expectation (mean) of the number of particles that will be detected in the next minute of testing?

3. Consider another detector with some unknown efficiency f, and a material with some unknown emission rate λ. The detector randomly detects each particle out of the total number emitted n_{emit} with probability f. Particles are initially emitted following a Poisson process.

   (a) What priors do you need to fully specify the problem?

   (b) Draw a probabilistic graphical model (also known as a Bayesian Network) showing the problem

   (c) The code here: http://bit.ly/2xHPOos runs a very simple and inefficient Metropolis-Hastings sampler. What priors does the the code assume?

(d) What distribution does the code model the number of detected particles n_{det} as following? Does this make sense? Why?

(e) Run the code to generate a Monte Carlo Markov Chain. Determine the mean value and 95% one-tailed upper limit of the emission rate lambda.

(6) [Hand-in to Sadegh Khochfar on Monday Nov 19]

(a) Starting with the density of states in six-dimensional phase space, derive the equation of state for a fully degenerate highly relativistic pure electron gas:

$$p = K\rho^{4/3},$$

where p is pressure, ρ is the total mass density, and K is a constant. Express K in terms of fundamental physical constants, and show how the result changes if neutrons are substituted for electrons.

How does p scale with ρ is the gas is non-relativistic?

(b) By comparing the degenerate equation of state with the ideal gas law, show that the degree of degeneracy of a partially degenerate non-relativistic neutron gas increases as a function of $\rho/T^{3/2}$, where T is the density and temperature of the gas.

(7) [Hand-in to Philip Best on Monday Nov 26]

(a) A diffraction grating is designed to operate at normal incidence at second order with a central wavelength of 650 nm, such that the angle of diffraction is 45°. Calculate the line density required in the grating.

(b) Over what range of wavelengths can the grating operate without confusion between spectral orders?

(c) If the aim is to achieve a spectral resolution of $R = \Delta\lambda/\lambda = 10000$, what minimum physical size of grating is required?

(d) The above setup is used to observe a star with an AB magnitude of 25 and a featureless spectrum, using an 8m telescope. Assuming spectral sampling at 2 pixels per FWHM, calculate the rate of arrival of photons per spectral pixel in the case of a perfectly efficient spectrograph and telescope (a flux density of 1 Jy corresponds to an AB magnitude of 8.90).

(e) In practice, the light to be dispersed is selected by placing a slit in front of the spectrograph. Provide a *qualitative* argument, on geometrical grounds, as to why this broadens the FWHM of the spectrum.

(8) [Hand-in to Ken Rice on Monday Jan 14]

(a) Assuming that a protostellar disc is isothermal and in vertical hydrostatic equilibrium, show that the vertical scale height is given by $h \sim c_s/\Omega$, where c_s is the sound speed, and Ω is the orbital angular frequency.

(b) At what particle size will a dust layer have a vertical thickness less than h in a disc with a surface density $\Sigma = 100$ g cm^{-2} and an viscous α of $\alpha = 10^{-3}$? How will this particle size vary with radius?

(c) What implication does the above have for the observation of protostellar accretion discs?

(d) How does the growth timescale via settling vary with orbital radius?

(e) Consider a disc with mass $M_{\rm disc} = \pi r^2 \Sigma$ and thickness h, at radius r from a star of mass M_*. By approximating the self-gravity of the disc as that of an infinite sheet, estimate the minimum Σ such that dominates the vertical acceleration of $z = h$. Hence, show that

$$\frac{M_{\rm disc}}{M_*} > \frac{h}{r},$$

is a rough condition for when self-gravity matters for the vertical structure.

(f) Also show that the result in (e) can be recast as the standard Toomre criterion

$$Q = \frac{c_s}{\Omega} \pi G \Sigma \sim 1.$$

(9) [Hand-in to Ross McLure on Monday Jan 21]

(a) Models of AGN accretion disks suggest that much of the X-ray radiation is liberated at a radius of $R \simeq 5R_s$, where R_s is the Schwarzschild radius. Ignoring relativistic effects and viscosity, show that if an object free-falls from infinity (and all its kinetic energy is released as radiation at $5R_s$) it will liberate $\sim 10\%$ of its rest mass energy.

(b) An AGN has a 0.5–2 keV X-ray luminosity of 10^{39} Watts. Use the Eddington limit to derive a lower limit to the mass of the central black hole. Assume a spectrum of $f_\nu \propto \nu^{-1}$ from the far-IR (100 microns) to hard X-rays (100 keV).

(c) Assuming the results of (a) and (b), what is the minimum time it would take a seed black hole (10 solar masses) in the centre of a proto-galaxy to grow by accretion at the Eddington rate to a size sufficient to power this quasar.

(10) [Hand-in to Beth Biller on Monday Jan 28]

Please see Figures 1 and 2.
(a) Show that the transit duration for a non-central transit is:

$$t_T = \frac{PR_*}{\pi a}\sqrt{\left(1+\frac{R_p}{R_*}\right)^2 - \left(\frac{a\cos i}{R_*}\right)^2}$$

where t_T is the total transit duration, R_* is the stellar radius, R_p is the planet radius, a is the orbital semi-major axis, and i is the orbital inclination (where i=90° is "edge-on").

(b) Show that the duration of the "flat part" of the transit light curve, i.e. the time when the planet is fully superimposed on the stellar disk is:

$$t_F = \frac{PR_*}{\pi a}\sqrt{\left(1-\frac{R_p}{R_*}\right)^2 - \left(\frac{a\cos i}{R_*}\right)^2}$$

(c) Describe why the impact parameter b is:

$$b = \frac{a}{R_*} \times \cos i$$

(d) Go to the following address: http://www.stefanom.org/systemic-live/ with your internet browser and click on 'Open Systemic'. After registration, you will be taken to the Systemic Console, where you will be carrying out the assignment. You can click on the question mark icons to open help pop-ups that explain the function of each panel. Note: if Systemic warns you that your browser might be too slow, please access the website using the Google Chrome browser (http://www.google.com/chrome) or update your browser. Follow along the tutorial at http://www.stefanom.org/51-pegged to fit your first planetary model. What is the final period and mass of the planet you found?

(e) HD 69830 is a multiple-planet system with three low-mass planets. Using the Console, find a three-planet fit to the data set that has the lowest possible chi-square. Print out the RV curve and the picture of the orbit. Write down the χ^2, RMS, and Jitter from the window on the right-hand side. Select the 'Dynamics' panel and integrate your fit for 1000 years using the stability checker. Is the system stable? Increase the mass of the middle planet to one Jupiter mass (this makes the chi-square go way up). Is this modified version of the system stable? Finally, compare your model planetary system to the published model: http://exoplanet.eu/star.php?st=HD+69830.

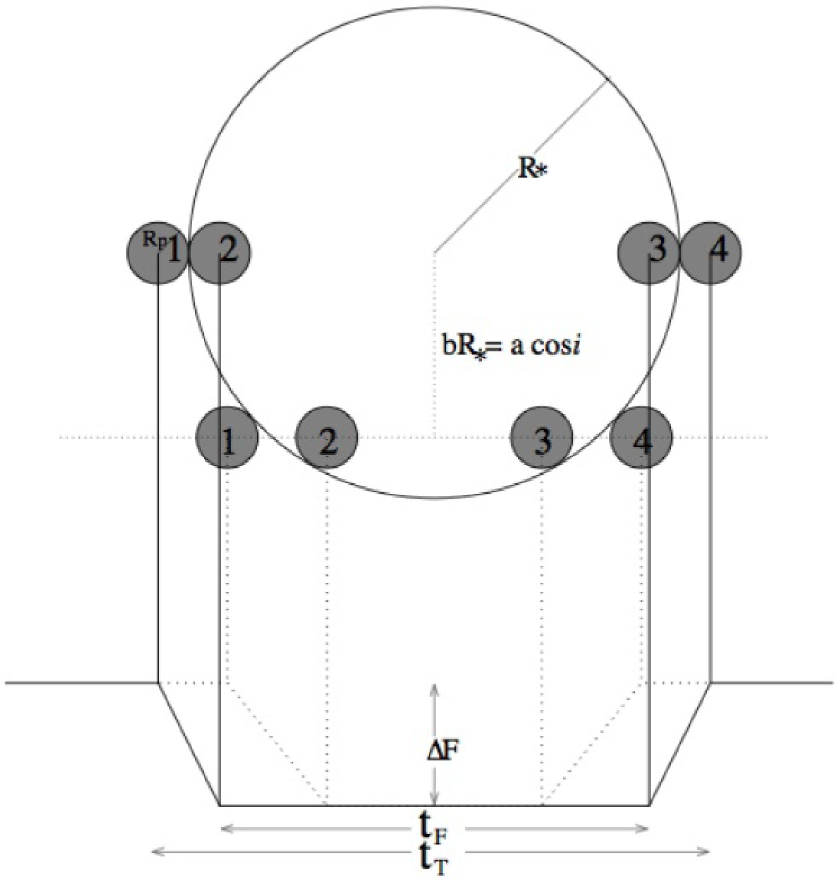

Figure 1: Definition of transit light-curve observables. Two schematic light curves are shown on the bottom (solid and dotted lines), and the corresponding geometry of the star and planet is shown on the top. Indicated on the solid light curve are the transit depth ΔF, the total transit duration t_T , and the transit duration between ingress and egress t_F (i.e., the flat part of the transit light curve when the planet is fully superimposed on the parent star). First, second, third, and fourth contacts are noted for a planet moving from left to right (not needed for this problem set). Also defined are R, R_p, and impact parameter b corresponding to orbital inclination i. Different impact parameters b (or different i) will result in different transit shapes, as shown by the transits corresponding to the solid and dotted lines.

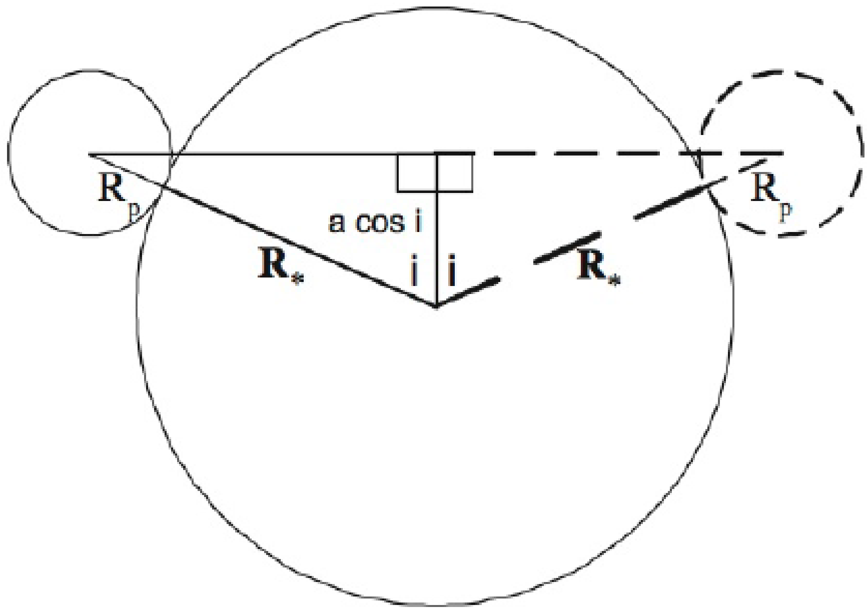

Figure 2: Transit planet schematic for deriving non-central transit parameters. Note the definition of i for orbital inclination ($i = 90°$ corresponds to edge-on). Figure from R. Santana.

(11) [Hand-in to Bob Mann on Monday Feb 4]

(a) Assuming that the nuclear factor $S(E)$ is a slowly-varying function of energy E, show that the reaction rate, $R_{\rm AB}$, for fusion of nuclei of types A and B, which is given by

$$R_{\rm AB} = n_{\rm A} n_{\rm B} \left(\frac{8}{\pi m_{\rm r}}\right)^{1/2} \left(\frac{1}{kT}\right)^{3/2} \int_0^\infty S(E) \exp\left[-\frac{E}{kT} - \left(\frac{E_{\rm G}}{E}\right)^{1/2}\right] {\rm d}E$$

has a maximum contribution at energy $E_0 = (E_{\rm G}(kT)^2/4)^{1/3}$.

(b) Sketch the integrand for $R_{\rm AB}$ and show, through use of a Taylor expansion, that the width of this peak, Δ, (also called the 'fusion window'), is given by

$$\Delta = \frac{4}{3^{1/2} 2^{1/3}} E_{\rm G}^{1/6} (kT)^{5/6}.$$

(c) The Gamow energy for nuclei of types A and B is given by $E_{\rm G} = (\pi \alpha Z_{\rm A} Z_{\rm B})^2 2 m_{\rm r} c^2$, where $Z_{\rm A}$, $Z_{\rm B}$ are their atomic numbers, respectively, $m_{\rm r} = m_{\rm A} m_{\rm B}/(m_{\rm A} + m_{\rm B})$ is their reduced mass, and $\alpha \simeq 1/137$ is the fine-structure constant.

Calculate the energy, E_0, in units of kT, at which most fusion reactions occur in the following three cases: (i) p-p chain reactions at the Sun's central temperature of $T = 1.5 \times 10^7$K; (ii) the CNO cycle under the same conditions; and (iii) helium burning at $T = 10^8$K.

(d) Show that, at constant $n_{\rm A}, n_{\rm B}$, the fusion rate per unit volume $R_{\rm AB}$ scales as T^β, where

$$\beta = \left(\frac{E_{\rm G}}{4kT}\right)^{1/3} - \frac{2}{3}$$

and compute the value of β for the three reaction systems above.

(12) [Hand-in to Avery Meiksin on Monday Feb 11]

The equations of stellar structure are:

$$\begin{aligned}
\text{(1) Equation of Continuity :} \quad \frac{dM(r)}{dr} &= 4\pi r^2 \rho(r) && (1)\\
\text{(2) Equation of Hydrostatic Equilibrium :} \quad \frac{1}{\rho}\frac{dP}{dr} &= -\frac{GM(r)}{r^2} && (2)\\
\text{(3) Equation of Energy Generation :} \quad \frac{dL}{dr} &= 4\pi r^2 \epsilon\rho && (3)\\
\text{(4) Equation of Radiative Diffusion :} \quad \frac{L}{4\pi r^2} &= -\frac{4ac}{3\rho\kappa} T^3 \frac{dT}{dr} && (4)
\end{aligned}$$

(a) Give the form of the equation of radiative diffusion in the case where the opacity is dominated by electron scattering. For what class of stars is this a good approximation?

(b) Assuming the perfect gas law, show that the scaling of the structure equations in this case implies the following relations between the total mass, total luminosity, radius, central density and central temperature: $\rho_c R^3 \propto M$ (from continuity); $\rho_c R^4 T_c \propto M^2$ (from hydrostatic equilibrium); $ML \propto T_c^4 R^4$ (from radiative diffusion). Hence show that stars dominated by electron scattering should obey $L \propto M^3$ independent of the mechanism of energy generation.

(13) [Hand-in to Andy Taylor on Monday Feb 18]

(a) Write down Friedmann's equation for the scale factor of the universe, $R(t)$. Assuming that the universe contains pressureless matter and vacuum energy only, rewrite the equation in terms of the density parameters $\Omega_{\rm m}$ and $\Omega_{\rm v}$.

(b) Hence show that if $\Omega_{\rm m} = 0.1$ and $\Omega_{\rm v} = 1.5$, then there could not have been a big bang.

What is the highest redshift we could expect to see in such a universe? (This will require some numerical experimentation.)

(c) The equation for a radial null geodesic in the Robertson–Walker metric is $dr = c\,dt/R(t)$.

Using the relation between redshift and scale factor, $1 + z \propto 1/R(t)$, plus Friedmann's equation, deduce the differential relation between comoving distance and redshift:

$$R_0\,dr = \frac{c}{H_0}\left[(1-\Omega)(1+z)^2 + \Omega_{\rm v} + \Omega_{\rm m}(1+z)^3 + \Omega_{\rm r}(1+z)^4\right]^{-1/2} dz.$$

(d) Integrate this expression for the case of the $\Omega = 1$ Einstein–de Sitter universe. In this model, calculate the apparent angle subtended by a galaxy of proper diameter 30 kpc, as a function of redshift (recall $c/H_0 = 3000\,h^{-1}$Mpc). Show that there is a critical redshift at which this angle has a minimum value.