White Dwarf Properties and the Degenerate Electron Gas

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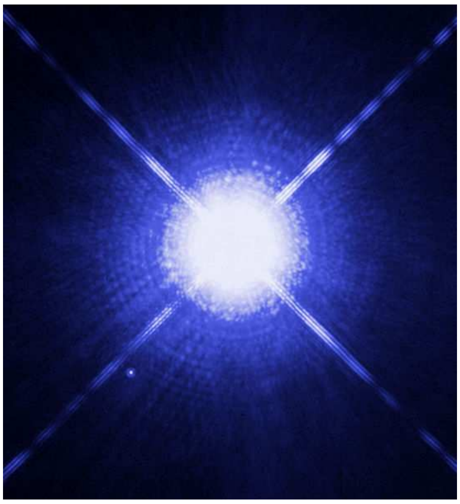

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1 Introduction

White dwarfs are the second most common type of star in the Galaxy, and represent the end stage of evolution for around 97% of all stars. Only those with masses greater than $\sim 8M_{\odot}$ will avoid this fate, and post nuclear-burning will become either neutron stars or black holes. Now devoid of the nuclear energy sources that drive the evolution of their progenitor stars, white dwarfs shine at the expense of their residual thermal energy. It takes many billions of years for this heat to radiate away into space, and as such white dwarfs contain an observable fossil record of star formation processes in the history of the Galaxy.

1.1 Discovery

The first two white dwarfs to be discovered were 40 Eridani B and Sirius B. Observations revealed these stars to be of a type fundamentally different to the 'ordinary' stars, and over several decades at the start of the twentieth century the theory of stellar structure was revised to incorporate this new class of star.

In particular, parallax measurements of 40 Eridani B showed it to be many magnitudes fainter than other stars of it's spectral type (figure 1). The existence of Sirius B was inferred from it's gravitational influence on it's companion, Sirius A. By analysis of the orbital dynamics, and an early measurement of gravitational redshift, it's mass could be measured. The observed mass and size of these stars implied a density several thousand times greater than anything observed before in nature, and the behaviour of material under these conditions was not well studied.

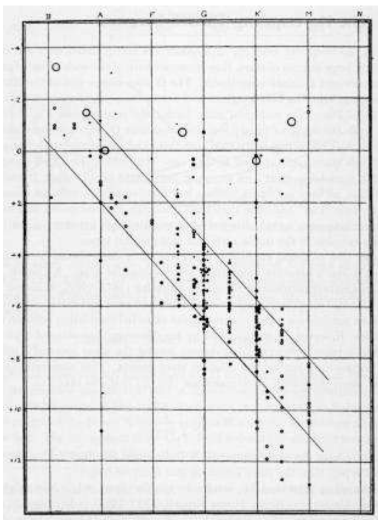

Figure 1: An early (1914) HR diagram showing 40 Eridani B at bottom left.

1.2 Survey Techniques

White dwarfs are typically around 10 magnitudes fainter than main sequence stars of the same colour. As such, they are observable only relatively close to the sun, and therefore exhibit large proper motions. This fact has historically been used to conduct surveys for white dwarfs through the use of the *reduced proper motion* statistic H (figure 2) which combines apparent magnitude in some band (m_B) and proper motion (μ) to estimate the intrinsic magnitude of stars:

$$H_B = m_B + 5\log(\mu) + 5$$

In this way Willem Luyten produced some of the first large proper motion catalogues containing ~ 3000 white dwarfs. Proper motions were detected and measured from photographic plates by eye using large 'blink comparators', which inevitably led to incompleteness problems due to objects being missed. Modern surveys avoid this by using automated search algorithms to pair up stars between observations taken at different epochs.

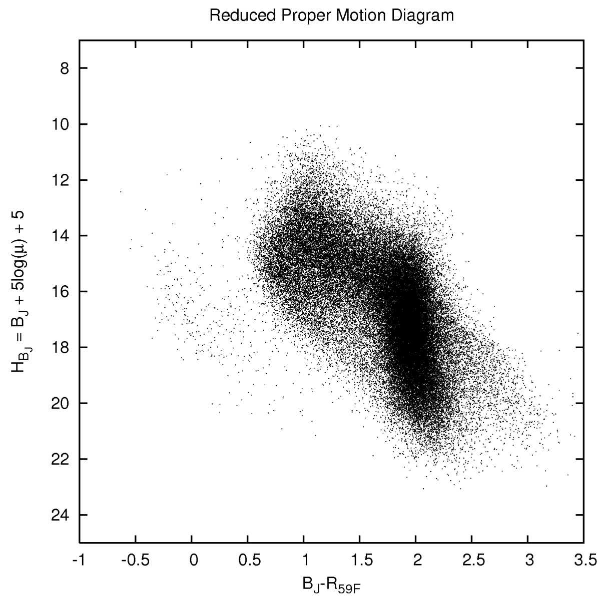

Figure 2: A reduced proper motion diagram showing the loci of various Galactic stellar populations and types.

Spectroscopic surveys can also reveal white dwarfs through their characteristically pressure-broadened absorption lines. For example, although SDSS primarily targets extragalactic objects, a large sample of white dwarfs has been spectroscopically observed due to their photometric colours merging with QSOs.

2 Sirius B - Internal Properties

Taking up to date values for the mass, radius and temperature of Sirius B from Holberg et al. [1998] we can infer some basic properties of the interior of white dwarf stars in general. It should be noted that Sirius B is not particularly representative of white dwarf stars, being of considerably higher mass, and is chosen more for historical context.

2.1 Pressure Ionisation

Holberg et al. quote a mass of $M \sim 1.05 M_{\odot}$ and radius of $R \sim 0.0084 R_{\odot}$, which leads to a mean density of $\sim 2.5 \times 10^{9}$ kg m^{-3}. Assuming an internal composition of pure carbon, this leads to a mean nuclear separation of $\sim 2 \times 10^{12}$ m. By contrast, the Bohr radius of a carbon ion with only one remaining electron is $\sim 8 \times 10^{12}$ m, and the interior must be entirely *pressure ionised.*

2.2 Electron Degeneracy

Holberg et al. derive an effective temperature of $T \sim 25000K$ for Sirius B. If we use this as a first order estimate of the interior temperature, we calculate a de Broglie wavelength of $\sim 6.85 \times 10^{-10}$ m for the (free) electrons, several orders or magnitude greater than the mean separation of 9.85×10^{-13} m. Therefore, any attempt to understand the internal structure must involve a quantum mechanical explanation. By contrast, the de Broglie wavelength of the ions, $\sim 4.63 \times 10^{-12}$ m, is only about twice the average separation. Therefore, to a good approximation we expect the ions to behave as an ideal classical gas.

3 The Equation of State for an Electron Gas

The equation of state for the degenerate interior of white dwarf stars can be derived from first principles by first considering the density of quantum states $g(p)$, defined in phase space as

$$g(p)\ \mathrm{d}p\mathrm{d}V = \frac{8\pi}{h^3} p^2\ \mathrm{d}p\mathrm{d}V.$$

This includes the degeneracy factor of two for electrons of opposite spin states. As electrons fermions, the distribution of particles amongst the quantum states obeys Fermi-Dirac statistics, which states that the average occupation of a state of energy ϵ is given by

$$f(\epsilon) = \frac{1}{exp[\frac{\epsilon-\mu}{kT}] + 1}.$$

Microscopically, pressure is defined as the flux of momentum through a unit surface. If we consider a surface element of area $\mathrm{d}\sigma$ with normal $\underline{\mathbf{n}}$, we can derive an expression for the

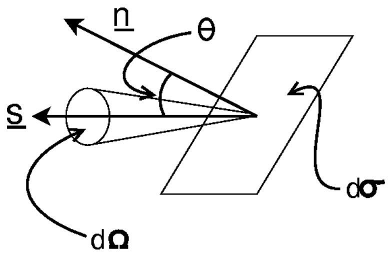

Figure 3: The equation of state is derived by considering the flux of momentum across surface element dσ in the direction **n**

pressure by considering how many electrons pass through it per second into an element of solid angle dΩ, in the direction **s**, in the momentum range $p \rightarrow p{+}\mathrm{d}p$ (figure 3).

The number density of states in the vicinity of dσ is given by $g(p)\mathrm{d}p$, and these are occupied according to $f(\epsilon)$. Each electron carries a momentum equal to $p\cos(\theta)$ in the direction **n**, and sees a projected surface area of $\cos(\theta)\mathrm{d}\sigma$. This is multiplied by the velocity of the electrons corresponding to the given momentum, expressed for now as $v(p)$. As the distribution function is isotropic, the fraction of electrons passing into dΩ is equal to $\frac{\mathrm{d}\Omega}{4\pi}$. Finally, to get the pressure on the surface, we divide out dσ and integrate over one hemisphere and all momentum states:

$$P = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{2\pi} \int_{p=0}^{\infty} \frac{8\pi p^3}{h^3} \frac{1}{exp[\frac{\epsilon(p)-\mu}{kT}]+1} \cos(\theta)^2 v(p)\mathrm{d}p \frac{\mathrm{d}\Omega}{4\pi}.$$

Integrating out the angular dependancy leaves:

$$P = \frac{4\pi}{3h^3} \int_{p=0}^{\infty} \frac{p^3}{exp[\frac{\epsilon(p)-\mu}{kT}]+1} v(p)\mathrm{d}p.$$

where $v(p)$ and $\epsilon(p)$ are the relativistic expressions for the velocity and energy of a given momentum state, respectively. μ has dimensions of energy and is called the *chemical potential*. In this context it is related to the degree to which the interior is degenerate. This formula cannot be solved analytically for all possible conditions in the white dwarf interior. However, there are several limiting cases in which this is possible, and these can be used to demonstrate certain properties that extend into the analytically intractible regimes.

3.1 Fully Degenerate Configurations

When all quantum states are occupied up to some momentum, and none above this, the electron gas is said to be *fully degenerate*. The momentum of the highest occupied states is denoted p_f and called the *Fermi momentum*, with the corresponding *Fermi energy* ϵ_f. This is equivalent to assuming a temperature of zero for the interior. This is obviously unphysical, and any thermal energy will promote electrons to higher momentum states. In reality, the interior of a white dwarf is only partially degenerate, and hydrostatic equilibrium is maintained by a complex mixture of degeneracy pressure and a small but finite thermal pressure. However, this assumption greatly simplifies the solution to the equation of state, as in this regime the distribution function takes the following form:

$$f(\epsilon) = \begin{cases} 1 & : \quad \epsilon \leq \epsilon_f \\ 0 & : \quad \epsilon > \epsilon_f \end{cases}$$

The pressure integral now becomes:

$$P = \frac{4\pi}{3h^3} \int_{p=0}^{p_f} p^3 v(p) \mathrm{d}p.$$

For the velocity, we rearrange the relativistic expression for momentum $p = \gamma(v)mv$ to obtain

$$v(p) = c \frac{\frac{p}{m_e c}}{\sqrt{(\frac{p}{m_e c})^2 + 1}}$$

which, when substituted into the above formula, allows us to write for the equation of state:

$$P = \frac{4\pi m_e^4 c^5}{3h^3} \int_{\xi=0}^{x} \frac{\xi^4}{\sqrt{1+\xi^2}} \mathrm{d}\xi$$

where we have used the substitutions $\xi = \frac{p}{m_e c}$ and $x = \frac{p_f}{m_e c}$. x is called the *relativity parameter* and appears widely in degeneracy pressure calculations. This integral has the following solution:

$$P = \frac{\pi m_e^4 c^5}{6h^3} \left[x(2x^2 - 3)\sqrt{(1+x^2)} + 3\sinh^{-1}(x) \right].$$

This is the general formula for the pressure at all values of the relativity parameter, for the fully degenerate case. Now, the number density of electrons can be expressed as a function of the Fermi momentum p_f by integrating the density of states in phase space over all possible values of the momentum ($p = 0 \rightarrow p_f$). This gives:

$$n_e = \frac{8\pi}{h^3} \int_{p=0}^{p_f} p^2 \mathrm{d}p = \frac{8\pi}{3h^3} p_f^3$$

which can be expressed in terms of x as

$$n_e = \frac{8\pi m_e^3 c^3}{3h^3} x^3$$

which allows one to express the pressure directly in terms of the number density of electrons. Before making this substitution, we may further simplify the equation of state by considering as two limiting cases the regimes of *non-relativistic degeneracy* ($x \to 0$) and *relativistic degeneracy* ($x \to \infty$). The parameter x is a measure of the momentum of the electrons in the highest populated energy states, and thus measures the importance of relativistic effects. In these two cases the general equation of state reduces to the following forms:

$$P \to \begin{cases} \frac{8\pi m_e^4 c^5}{30h^3} x^5 & x \to 0 \\ \\ \frac{\pi m_e^4 c^5}{3h^3} x^4 & x \to \infty \end{cases}$$

Now substituting in the number density of electrons:

$$P = \frac{1}{40} \left(\frac{3}{\pi} \right)^{\frac{2}{3}} \frac{h^2}{m_e} n_e^{\frac{5}{3}}$$

$$P = \frac{1}{16} \left(\frac{3}{\pi} \right)^{\frac{1}{3}} hcn_e^{\frac{4}{3}}$$

These expressions give approximations to the equation of state for completely degenerate stellar configurations, in the limiting regimes of non-relativistic and relativistic degeneracy.

3.2 Consequences for the Mass of White Dwarfs

Putting the equation of state for the non-relativistic, fully degenerate white dwarf interior into the equation for hydrostatic equilibrium yields the following approximate result:

$$P \sim \left(\frac{M^2}{R^4} \right) \sim \left(\frac{M}{R^3} \right)^{\frac{5}{3}}$$

which has the solution:

$$R \sim \frac{1}{M^{\frac{1}{3}}}.$$

Thus more massive white dwarfs are expected to be *smaller*, a feature not observed in main sequence stars. Also, the relativistic 'softening' of the equation of state has profound implications for the *maximum* mass of white dwarf stars, as first noted by Chandrasekhar in 1931. Specifically, the reduction in stiffness gives rise to a *limiting mass* for highly relativistic white dwarfs. Repeating the above argument for the relativistic-degenerate configuration gives the relation:

$$P \sim \left(\frac{M^2}{R^4} \right) \sim \left(\frac{M}{R^3} \right)^{\frac{4}{3}}$$

$$\frac{M^{\frac{6}{3}}}{R^4} \sim \frac{M^{\frac{4}{3}}}{R^4}$$

R cancels out, implying the existence of a unique mass for relativistic white dwarfs, above which hydrostatic equilibrium cannot be maintained and the star starts to collapse. This unique mass is called the Chandrasekhar Mass, and has the definition given by Chandrasekhar [1939]:

$$M_{Ch} = 5.75\mu_e^{-2}M_\odot$$

where μ_e is the number of nucleons per electron, and is a measure of the metallicity of the interior. M_{Ch} has the value $1.44M_\odot$ for a composition of pure He^4.

3.3 Thermal and Mechanical Properties Decoupled

It was shown earlier that the ions present in the interior of the star behave, to a good approximation, like an ideal classical gas, and the electrons as an ideal Fermi gas. How does the pressure in each component contribute to hydrostatic support? Using data for Sirius B discussed in section 2, we can estimate the pressure of the ions and electrons separately. The equation of state for the ions is $P_i = n_i k_B T$, and for the electrons $P_e = \frac{1}{40}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} \frac{h^2}{m_e} n_e^{\frac{5}{3}}$, where $n_e = 12n_i$ for an internal composition of pure carbon. Putting the numbers in leads to the following ratio for the relative magnitude of the pressures:

$$\frac{P_e}{P_i} = 2.148 \times 10^{-18} n_i^{\frac{2}{3}}.$$

Given the mean density of Sirius B, one calculates an ion number density of $n_i = 1.25\times10^{35}\text{m}^{-3}$. Putting this into the above ratio gives:

$$\frac{P_e}{P_i} = 5.37 \times 10^5.$$

Thus the mechanical properties of white dwarfs are completely dominated by the degenerate electron gas.

The thermal properties can be explored by considering the heat capacities of the two components, defined at constant volume as $C_v = \frac{\partial U}{\partial T}\Big|_v$. The internal energy of both the ideal classical gas of ions and ideal Fermi gas of electrons satisfies $U \propto P$. However, the equation of state for the electrons has no T dependency in the fully degenerate limit, and the total combined heat capacity of the interior is found to be:

$$\begin{aligned} C_v &= \frac{\partial U_i}{\partial T}\bigg|_v + \frac{\partial U_e}{\partial T}\bigg|_v \\ &= \frac{3}{2} n_i k \end{aligned}$$

The heat capacity of the degenerate electrons is equal to zero; the thermal energy content of the star is dominated by the ions.

References

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