Quantum Mechanics 3   2001/2002

Solution set 1

(1)

(a) Must have $\psi(L) = \psi(-L) = 0$, so $B\cos kL + A\sin kL = 0$ and $B\cos kL - A\sin kL = 0$. Add or subtract these, to get $B = 0$ and $\sin kL = 0$, or $A = 0$ and $\cos kL = 0$.

(b) This is easier, since $\psi(0) = 0$, so the cos solution is never involved and $\psi = A\sin kx$, with $k = n\pi/(2L)$, $n = 1, 2, 3, \ldots$

(c) We need $\int_{-L}^{L} |A|^2 \cos^2 kL\, dx = 1$ for odd n. Since $\cos^2 y = (1+\cos 2y)/2$, and the cos part averages to zero, this gives $|A| = 1/L^{1/2}$ (the phase of A is arbitrary). The working for even n is similar.

(d) By symmetry, $\langle x\rangle = 0$ (if we use the symmetric convention, where the well runs from $-L$ ro L). The rms (or standard deviation) is the square root of the variance, which is $\langle x^2\rangle - \langle x\rangle^2$; for the symmetric convention, we don't need the second term. For the second moment, we need $\langle x^2\rangle = (1/L)\int_{-L}^{L} x^2 \sin^2(n\pi x/2L)\, dx$ (even n), or the same with $\sin \to \cos$ (odd n). Integration by parts gives

$$\langle x^2\rangle = \frac{L^2}{3}\left(1 - \frac{6}{n^2\pi^2}\right)$$

for any n. For the classical limit, $\langle x^2\rangle = (1/2L)\int_{-L}^{L} x^2\, dx$.

(e) Inside the well, $V = 0$, so $E = \hbar^2 k^2/2m$, with k given by the boundary conditions.

(f) For the second part, set $\hbar\omega$ ($= 2\pi\hbar c/\lambda$) to the difference in energy between the $n = 1$ and $n = 2$ states ($= 3\pi^2\hbar^2/[8mL^2]$). The effective size is therefore

$$L = \sqrt{\frac{3\pi\hbar\lambda}{16mc}} \simeq 1.66 \times 10^{-10}\,\mathrm{m}.$$

(2)

(a) Stationary state: $\psi(x,t) = u(x)e^{-i\omega t}$, where $\omega = E/\hbar$. Similarly, $\psi^*(x,t) = u^*(x)e^{+i\omega t}$. Therefore, the $e^{\pm i\omega t}$ terms cancel in $|\psi|^2 = \psi\psi^*$, leaving ρ independent of t. The same happens with $\mathbf{j}$.

If $\dot{\rho} = 0$, the given equation of continuity says $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$. This doesn't require $\mathbf{j}$ to be a constant vector, but in 1D this is just $dj/dx = 0$, so j is constant.

(b) For a plane wave, $\boldsymbol{\nabla}\psi = i\mathbf{k}\psi$. Also, $\boldsymbol{\nabla}\psi^* = (\boldsymbol{\nabla}\psi)^* = -i\mathbf{k}\psi^*$. Thus $\mathbf{j} = (\hbar/m)\mathbf{k}|\psi|^2$. Now, $\hbar k/m$ is momentum over mass, or velocity. So, j is probability density times velocity, reasonably enough if j is to be a flux density of probability.