Quantum Mechanics 3 2001/2002

Problem set 6

(1) [1996 resit exam] The operator a is defined by

$$a \equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2} x + \frac{i}{(2m\omega\hbar)^{1/2}} p.$$

Given that the one-dimensional simple harmonic oscillator is described by the Hamiltonian

$$H = \hbar\omega(a^\dagger a + 1/2),$$

and that $[a, a^\dagger] = 1$, show that $[H, a] = -\hbar\omega a$ and that $[H, a^\dagger] = \hbar\omega a^\dagger$.

Use these commutation relations to establish the lowering and raising properties of a and $a^\dagger$ respectively, and explain carefully how the energy eigenvalue spectrum may be found.

(2) [2000 resit exam] The raising and lowering operators for the one-dimensional harmonic oscillator are respectively

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\alpha x - \frac{i}{\alpha\hbar} p\right)$$

and

$$a = \frac{1}{\sqrt{2}}\left(\alpha x + \frac{i}{\alpha\hbar} p\right),$$

where $\alpha = \sqrt{m\omega/\hbar}$ and p is the momentum operator.

(a) Use these operators to derive the normalized ground-state wave function, $u_0(x)$ (consider the effect of a and $a^\dagger$ on u_0).

(b) Similarly, derive the first excited state, $u_1(x)$, but do not normalize it.

(c) Show that the state u_0 satisfies the Schrödinger equation, and has the expected energy.