Quantum Mechanics 3   2001/2002

Problem set 5

(1) [Long question, 1999 resit paper]

(a) Define the angular momentum operators L_x, L_y, L_z in terms of the position and momentum operators. Prove the following commutation result for these operators: $[L_x, L_y] = i\hbar L_z$.

(b) Show that the operators $L_\pm = L_x \pm i L_y$ act as raising and lowering operators for the the z component of angular momentum, by first calculating the commutator $[L_z, L_\pm]$.

(c) A system is in the state ψ, which is an eigenstate of the operators L^2 and L_z, with quantum numbers ℓ and m. Calculate the expectation values $\langle L_x \rangle$ and $\langle L_x^2 \rangle$ (hint: express L_x in terms of $L_\pm$).

(d) Hence show that L_x and L_y satisfy a general form of the uncertainty principle: $\langle (\delta A)^2 \rangle \langle (\delta B)^2 \rangle \geq -\langle [A, B] \rangle^2 / 4$.

(2) Using the commutator $[L_x, L_y] = i\hbar L_z$ and its cyclic variants, prove that total angular momentum squared and the individual components of angular momentum are compatible variables (i.e. $[L^2, L_x] = 0$ etc.).

(3) Write down the definitions of the operators L_x, L_y and L_z in Cartesian coordinates. Show that, in spherical polars, the operators become

$$
\begin{aligned}
L_x / i\hbar &= \sin\phi \frac{\partial}{\partial\theta} + \cot\theta \, \cos\phi \frac{\partial}{\partial\phi} \\
L_y / i\hbar &= -\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \, \sin\phi \frac{\partial}{\partial\phi} \\
L_z / i\hbar &= -\frac{\partial}{\partial\phi}.
\end{aligned}
$$

(remember cot = 1/tan).

Look up the full form of ∇^2 in your maths notes, and hence show that L^2 is proportional to the angular part of ∇^2:

$$
L^2 \psi = -\hbar^2 r^2 \left[\nabla^2 \psi - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) \right].
$$